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LINEARIZED KAPPA GUIDANCE

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An alternative solution to the κ-guidance dynamical equations is presented in this report. κ-guidance is a midcourse guidance algorithm which maximizes intercept velocity. The nonlinear κ-guidance dynamical equations are linearized. The linearization is via a nonlinear coordinate transformation and nonlinear feedback. The cost function is approximated by a quadratic cost. The linear quadratic regulator (LQR) method is then used to solve the optimal control problem. A closed form solution to the LQR problem is derived for both the free and fixed final angle of approach cases. A simulation is conducted to compare the trajectories and costs of the linearized solution method and the original solution method. The results show similar performance for the two methods.

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FOREWORD

This report presents an alternate method to the solution of the kappa guidance optimal control problem. Kappa guidance is a method of computing midcourse guidance for missiles, e.g., surface to air missiles. The kappa guidance optimal control problem comprises the state dynamical equations and the cost function. The cost function is set to maximize the probability of kill. The objective of optimal control is to solve the state dynamical equations while minimizing the cost function. The original published method, in Reference [2] to this report, gives a suboptimal feedback controller. This controller generates a solution to the guidance equations so trajectories may be computed. It is suboptimal since approximations are made in its derivation. The alternate method presented is referred to as linearized kappa guidance since a linearizing coordinate transformation and nonlinear feedback are used on the nonlinear state dynamical equations. The linearized method is also suboptimal, as approximations are made in its solution; however, the approximations are different from those used in the original method. The main feature of the linearized method is that closed form solutions for the states are derived. Such closed form solutions for the states were not obtained in the original derivation. The costs of trajectories computed by each method for identical initial conditions are comparable.

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CONTENTS

ILLUSTRATIONS

CHAPTER 1

INTRODUCTION

In this report, an alternate solution method to the κ guidance optimal control problem is presented. Kappa guidance is a technique for computing the midcourse guidance of a missile; e.g., a surface to air missile. Kappa guidance optimizes the midcourse trajectory of a missile so that the terminal velocity is maximized. This increases the probability of kill. The current solution method for the κ guidance optimal control problem (which we will also refer to as the original method) is to write the Hamiltonian and obtain a solution from the necessary conditions. Closed form feedback controllers are obtained for both the free final angle of approach case and the fixed final angle of approach case with the original method. The original solution method is suboptimal because approximations are made in its derivation. In this alternate method the nonlinear dynamics, which also vary with respect to the independent parameter range, are linearized via a nonlinear coordinate transformation and nonlinear feedback. The cost function, which is nonlinear, is still nonlinear in the transformed coordinates. A quadratic approximation to the cost function is made in the transformed coordinates. Then the (transformed) guidance problem is solved using Linear Quadratic Regulator (LQR) control on the linearized dynamics and approximate quadratic cost. Closed form state equations and adjoint variable equations are obtained. That is, the velocity vector angle and heading error angle for the midcourse trajectory are known in closed form with this alternate method. Because of the cost approximation, the alternate solution method is also suboptimal. The main feature of the alternate method presented here is that closed form solutions for the states are derived. Such closed form solutions were not obtained with the original method. The costs of trajectories computed by each method for identical initial conditions are comparable.

In Chapter 2, the κ guidance optimal control problem will be reviewed. A detailed discussion of κ guidance and the original method is given in Lin.[1] In Chapter 3 the dynamics of the problem are linearized using a nonlinear coordinate transformation and nonlinear feedback. The LQR optimal control problem is set up in Chapter 4. The necessary conditions are given and solutions considered for free and fixed final angles of approach in Chapter 5. Examples comparing the original and alternate solutions are given in Chapter 6. Much of this paper was presented in Serakos-Lin.[2]

CHAPTER 2

DYNAMICAL EQUATIONS

In this chapter, the dynamical (state) equations and optimal control problem will be reviewed. The κ guidance dynamical equations are

$$\frac{d}{dP}\begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \kappa \sec(\delta) \\ \kappa \sec(\delta) + \tan(\delta)/(R_0 - P) \end{bmatrix}, \tag{2.1}$$

where the state variables γ and δ are the velocity vector angle and the heading error angle respectively. The range to the projected intercept point (PIP) is represented by R, and κ is the curvature. The initial range to the target is R_0, and we define $P = R_0 - R$. The independent parameter in this problem is taken to be P. It is interesting to note that the derivative of P is dependent on the state variables since $\dot{P}$ is positive or negative depending on whether $-\pi/2 < \delta < \pi/2$ or $\pi/2 < \delta < -\pi/2$. (In many engineering problems, the independent variable is time, which generally does not depend on the state variables.) Under every normal circumstance, however, $-\pi/2 < \delta < \pi/2$. The curvature κ is also the control. The cost is

$$\mathcal{J}(\gamma, \delta, P, \kappa) = \int_0^{R_0} \left[(\kappa^2/2) + \omega^2\right] \sec(\delta)\, dP , \tag{2.2}$$

where ω is a constant which depends on aerodynamic parameters; see Equation (8-150) of Lin.[1] This original formulation was developed by F. Reifler.[3] Figure 2-1 shows the relationships between the states. The state equations, Equation (2.1), may be derived from the geometry and the definition of κ. The argument of Equation (2.2) contains κ^2, which penalizes excessive curvature, and $\sec(\delta)$, which penalizes large heading error angle. These work to maximize intercept velocity, which increases the probability of kill. Hence, minimizing the cost $\mathcal{J}(\gamma, \delta, P, \kappa)$ in Equation (2.2), maximizes the intercept velocity. The discrepancy of these state and cost equations with those given in Lin,[1] Gray-Hecht,[4] Ohlmeyer[5] and Serakos-Lin[2] is discussed in Appendix A (e.g., the minus signs on the right-hand side of Equation (2.1)).

We rewrite the dynamical equations to be in a form that is more familiar to control engineers. Let $x_1 = \gamma$ and $x_2 = \delta$. Then, Equation (2.1) becomes

$$\frac{d}{dP}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \tan(x_2)/R \end{bmatrix} + \sec(x_2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \kappa . \tag{2.3}$$

The variables x_1 and x_2 are referred to as the state variables. Equation (2.3) represents the *plant.* Let

$$f(x_1, x_2, R) = \begin{bmatrix} 0 \\ \tan(x_2)/R \end{bmatrix}, \qquad g(x_1, x_2, R) = \sec(x_2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} . \tag{2.4}$$

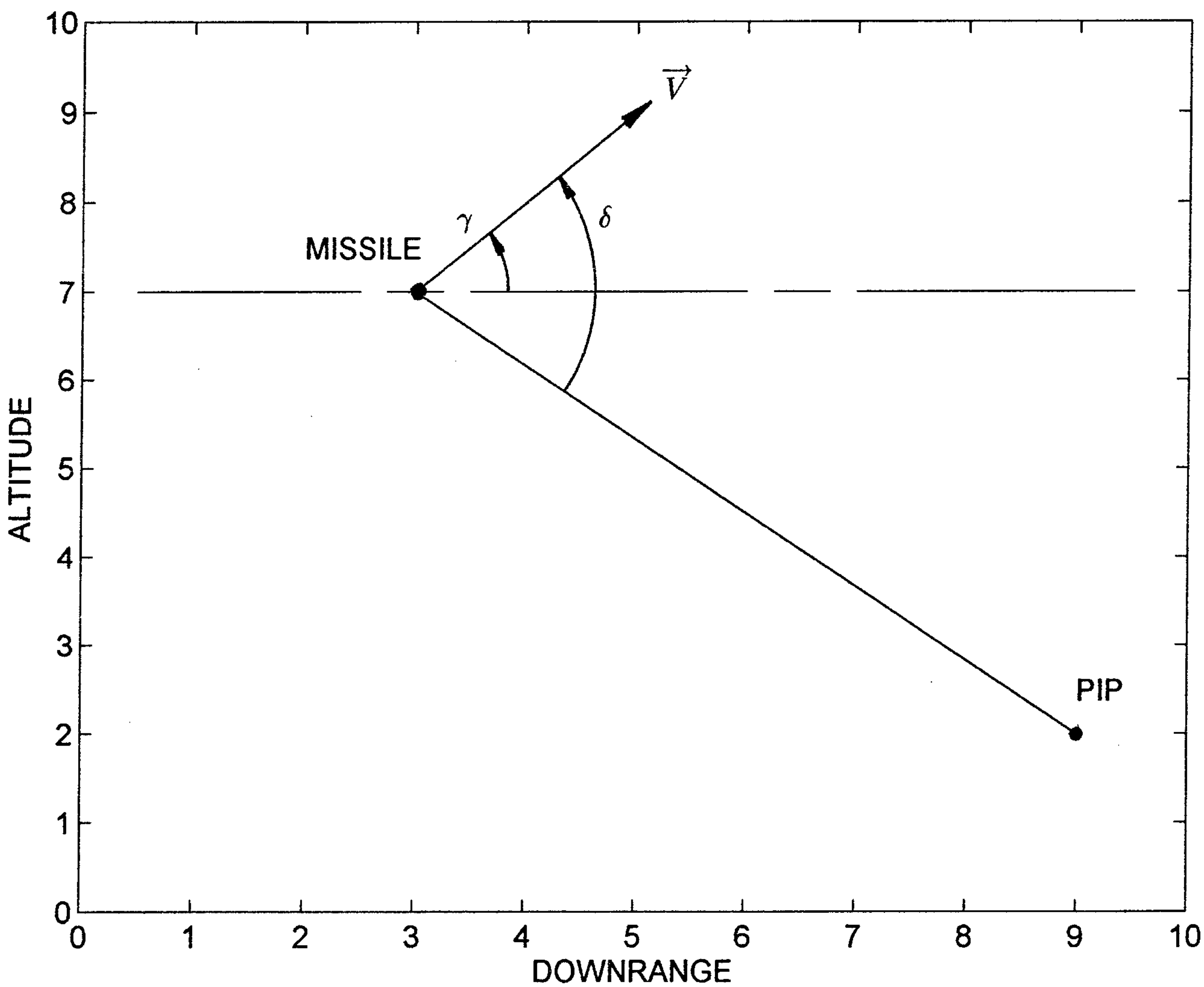

FIGURE 2-1. MID-COURSE GUIDANCE

Then, Equation (2.3) becomes

$$\frac{d\vec{x}}{dP} = f(x_1, x_2, R) + g(x_1, x_2, R) \cdot \kappa \ . \tag{2.5}$$

Equations (2.5) and (2.2) represent the κ midcourse guidance optimal control problem. The original method proceeded to solve this problem by obtaining the Hamiltonian and solving for the necessary conditions. A detailed discussion of the original method is presented in Lin,[1] Section 8.6.3. The state equations are given in Lin[1] by Equations (8-140a,b), and the cost is given by Equation (8-150). (As previously stated, the discrepancy is discussed in Appendix A.) The closed form solution for the feedback control in Lin[1] is given by Equation (8-167) in the free final angle of approach case, and by Equation (8-165) when the final angle of approach is specified. The original solution method is suboptimal since approximations are made in obtaining these controllers; see section 8.6.3 of Lin.[1]

CHAPTER 3

LINEARIZING COORDINATE TRANSFORMATION AND FEEDBACK

In this chapter, we will find a nonlinear coordinate transformation and feedback that will linearize Equation (2.5), the state equation. Consider the coordinate transformation

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = F(x_1, x_2, R) = \begin{bmatrix} x_1 - x_2 \\ -\tan(x_2)/R \end{bmatrix} . \tag{3.1}$$

Note that the transformation exists for all $R \neq 0$. Although the independent variable is P, in many cases it will be more convenient to work with R $(= R_0 - P)$. The inverse transformation is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = F^{-1}(z_1, z_2, R) = \begin{bmatrix} z_1 - \tan^{-1}(R\,z_2) \\ -\tan^{-1}(R\,z_2) \end{bmatrix} . \tag{3.2}$$

Define a nonlinear feedback by

$$\kappa = \alpha(x_1, x_2, R) + \beta(x_1, x_2, R) \cdot u \ , \tag{3.3}$$

where

$$\alpha(x_1, x_2, R) = -\frac{\sin(x_2)(\cos^2(x_2) + 1)}{R} \tag{3.4}$$

and

$$\beta(x_1, x_2, R) = -R\,\cos^3(x_2) \ , \tag{3.5}$$

and where u is the input to the linearized closed loop system. (The coordinate transformation and nonlinear feedback represented by Equations (3.1) and (3.3) where derived using the techniques from the differential geometric approach to nonlinear control, see Isidori[6]). Using the coordinate transformation and the nonlinear feedback, we compute the representation of the closed loop system in the $\overrightarrow{z}$ coordinates. The chain rule gives

$$\frac{d\overrightarrow{z}}{dP} = \frac{\partial F}{\partial \overrightarrow{x}} \frac{d\overrightarrow{x}}{dP} + \frac{\partial F}{\partial R} \frac{dR}{dP} .$$

Denote

$$F_* \triangleq \frac{\partial F}{\partial \overrightarrow{x}} = \begin{bmatrix} 1 & -1 \\ 0 & -\sec^2(x_2)/R \end{bmatrix} . \tag{3.6}$$

The closed loop system (Equation (2.5) with the nonlinear feedback Equation(3.3)) is

$$\frac{d\overrightarrow{z}}{dP} = F_*(f + g\alpha)(\overrightarrow{x}) + F_*(g\beta)(\overrightarrow{x}) \cdot u + \frac{\partial F}{\partial P}(\overrightarrow{x}) . \tag{3.7}$$

Computing, we have that

$$F_*(f+g\alpha)(\overrightarrow{x}) = \begin{bmatrix} 1 & -1 \\ 0 & -\sec^2(x_2)/R \end{bmatrix} \begin{bmatrix} 0 \\ \tan(x_2)/R \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & -1 \\ 0 & -\sec^2(x_2)/R \end{bmatrix} \cdot \sec(x_2) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \left(-\frac{\sin(x_2)(\cos^2(x_2)+1)}{R} \right)$$

$$= \begin{bmatrix} -\tan(x_2)/R \\ -\sec^2(x_2)\tan(x_2)/R^2 \end{bmatrix} + \begin{bmatrix} 0 \\ \sec^3(x_2)\sin(x_2)(\cos^2(x_2)+1)/R^2 \end{bmatrix} = \begin{bmatrix} -\tan(x_2)/R \\ \tan(x_2)/R^2 \end{bmatrix} .$$

Hence,

$$F_*(f+g\alpha)(\overrightarrow{z}) = \begin{bmatrix} z_2 \\ -z_2/R \end{bmatrix} .$$

Also,

$$F_*(g\beta)(\overrightarrow{x}) = \begin{bmatrix} 1 & -1 \\ 0 & -\sec^2(x_2)/R \end{bmatrix} \cdot \sec(x_2) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot (-R\cos^3(x_2)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

And

$$\frac{\partial F}{\partial P}(\overrightarrow{x}) = \frac{\partial F}{\partial R}(\overrightarrow{x})\frac{dR}{dP} = -\begin{bmatrix} 0 \\ \tan(x_2)/R^2 \end{bmatrix} .$$

(Note that $dR/dP = -1$.) So,

$$\frac{\partial F}{\partial P}(\overrightarrow{z}) = \begin{bmatrix} 0 \\ z_2/R \end{bmatrix} .$$

Hence, the closed loop system in $\overrightarrow{z}$ coordinates is

$$\frac{d}{dP}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u . \tag{3.8}$$

A block diagram representing the situation given by Equations (2.3), (2.2), (3.1) and (3.3) is given in Figure 3-1. It is interesting to note that the right-hand side of Equation (3.7) depends on P while the right side of Equation (3.8) does not. Apparently, the coordinate transformation F and nonlinear feedback α and β vary with respect to P, in such a way that the dependence on P in Equation (3.8) cancels out.

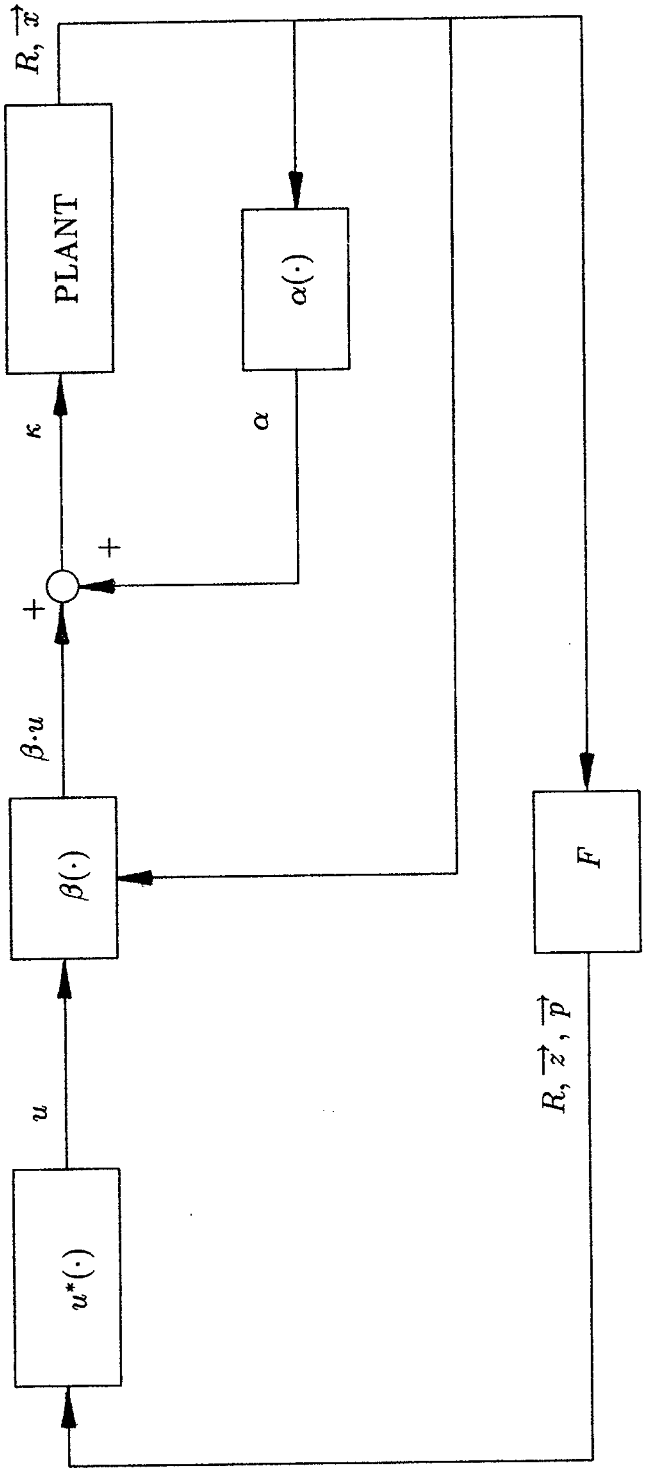

FIGURE 3-1. LINEARIZED KAPPA GUIDANCE

CHAPTER 4

LQR OPTIMAL CONTROL PROBLEM FORMULATION

In this chapter we will set up the LQR optimal control problem. The cost function, Equation (2.2), is not in the desired quadratic form after the nonlinear feedback and coordinate transformation. Hence, a quadratic approximation to the cost will be made. Along with the linearized dynamics, Equation (3.8), this will complete the setup of the LQR optimal control problem.

The Taylor series expansions for α and β are

$$\alpha = -\frac{2}{R}x_2 + \frac{4}{3R}x_2^3 + \cdots \text{ and } \qquad \beta = -R + \frac{3}{2}Rx_2^2 + \cdots .$$

Hence, for small x_2 and u

$$\begin{aligned} \frac{\kappa^2}{2} &= \frac{\alpha^2}{2} + \alpha\beta u + \frac{\beta^2 u^2}{2} \\ &\approx \frac{2 \cdot x_2^2}{R^2} + 2 \cdot x_2 u + \frac{R^2 u^2}{2} . \end{aligned} \tag{4.1}$$

For small x_2, the secant may be approximated by

$$\sec(x_2) = \frac{1}{\cos(x_2)} \approx \frac{1}{(1 - \frac{1}{2}x_2^2)} \cdot \frac{(1 + \frac{1}{2}x_2^2)}{(1 + \frac{1}{2}x_2^2)} \approx 1 + \frac{1}{2}x_2^2 . \tag{4.2}$$

Since this is a *quadratic* approximation, x_2 and u do not have to be as small as they would have to be if a linear approximation were being made. This point will be further considered in the examples. From Equations (4.1) and (4.2) we have

$$\begin{aligned} (\frac{\kappa^2}{2} + \omega^2)\sec(x_2) &\approx (\frac{2 \cdot x_2^2}{R^2} + 2 \cdot x_2 u + \frac{R^2 u^2}{2} + \omega^2) \cdot (1 + \frac{1}{2}x_2^2) \\ &\approx (\frac{2}{R^2} + \frac{\omega^2}{2})x_2^2 + 2 \cdot x_2 u + \frac{R^2}{2}u^2 + \omega^2 . \end{aligned} \tag{4.3}$$

The "$+\omega^2$" term in the right hand side of Equation (4.3) may be deleted without changing the optimal control problem; hence, the cost may be approximated by

$$\mathcal{J} \approx \int_0^{R_0} \left\{ (\frac{2}{R^2} + \frac{\omega^2}{2})x_2^2 + 2 \cdot x_2 u + \frac{R^2}{2}u^2 \right\} dP . \tag{4.4}$$

(Later, when a numerical value for the cost of a particular trajectory is computed, the "$+\omega^2$" term will be put back in.) The coordinate transformation and its inverse, Equations (3.1) and (3.2), may similarly be approximated,

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \approx \begin{bmatrix} x_1 - x_2 \\ -x_2/R \end{bmatrix} ; \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \approx \begin{bmatrix} z_1 - z_2 R \\ -z_2 R \end{bmatrix} . \tag{4.5}$$

In $\overrightarrow{z}$ coordinates, the cost may be approximated by substituting Equation (4.5) into Equation (4.4) to get

$$\mathcal{J} \approx \int_0^{R_0} \left\{ (2 + \frac{R^2\omega^2}{2}) z_2^2 - 2 \cdot R z_2 u + \frac{R^2}{2} u^2 \right\} dP \, . \tag{4.6}$$

The LQR optimal control problem is given by Equations (3.8) and (4.6). The solution to this LQR optimal control problem would be an approximate solution to the original optimal control problem given by Equations (2.1) and (2.2).

CHAPTER 5

NECESSARY AND SUFFICIENT CONDITIONS

In this chapter the necessary conditions and their solution, for both the free and fixed final approach angle cases, will be considered. For convenience we repeat the state, or plant, equations given by Equation (3.8),

$$\frac{d\overrightarrow{z}}{dP} = A\overrightarrow{z} + B \cdot u \ , \tag{5.1}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \tag{5.2}$$

and cost function given by Equation (4.6),

$$\overline{\mathcal{J}} = \int_0^{R_0} \left\{ (2 + \frac{R^2\omega^2}{2})z_2^2 - 2 \cdot Rz_2u + \frac{R^2}{2}u^2 \right\} dP \ , \tag{5.3}$$

for the LQR optimal control problem being considered. The solution to this form of the linearized quadratic problem is well known. See Athans-Falb.[7] However, we will go through the solution as a review. This will show the formulation of the optimal control problem, the necessary conditions and the Riccati equation.

FORMULATION OF HAMILTONIAN AND OPTIMAL CONTROL

The Hamiltonian is (see Athans-Falb[7])

$$\mathcal{H}(\overrightarrow{z}, \overrightarrow{p}, u, R) \doteq \mathcal{L}(\overrightarrow{z}, u, R) + \left\langle \overrightarrow{p}, f(\overrightarrow{z}, u) \right\rangle \ , \tag{5.4}$$

where

$$\mathcal{L}(\overrightarrow{z}, u, R) \doteq (2 + \frac{R^2\omega^2}{2})z_2^2 - 2 \cdot Rz_2u + \frac{1}{2}R^2u^2 \tag{5.5}$$

and

$$f(\overrightarrow{z}, u) \doteq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = \begin{bmatrix} z_2 \\ u \end{bmatrix} . \tag{5.6}$$

The vector $\overrightarrow{p} = (p_1, p_2)'$ contains the adjoint variables. Substituting Equations (5.5) and (5.6) into Equation (5.4) gives for the Hamiltonian,

$$\mathcal{H} = (2 + \frac{R^2\omega^2}{2})z_2^2 - 2 \cdot Rz_2u + \frac{1}{2}R^2u^2 + p_1z_2 + p_2u \ . \tag{5.7}$$

Now, from the minimum principle, we obtain two pairs of differential equations which the state and adjoint variables must satisfy. The first pair are the state equations, which we obtain from

$$\frac{d\vec{z}}{dP} = \frac{\partial \mathcal{H}}{\partial \vec{p}} \; . \tag{5.8}$$

The second pair are the adjoint equations,

$$\frac{d\vec{p}}{dP} = -\frac{\partial \mathcal{H}}{\partial \vec{z}} = -\begin{bmatrix} 0 \\ p_1 + (4 + R^2\omega^2)z_2 - 2 \cdot Ru \end{bmatrix} . \tag{5.9}$$

The optimal control must minimize the Hamiltonian; hence, $\partial\mathcal{H}(u)/du = 0$. (This is a necessary condition.) We differentiate Equation (5.7) to find the extremum of $\mathcal{H}$ with respect to u,

$$\frac{\partial \mathcal{H}}{\partial u} = -2 \cdot Rz_2 + p_2 + R^2 u = 0 \; . \tag{5.10}$$

Solving for u,

$$u^* = \frac{2z_2}{R} - \frac{p_2}{R^2} \, , \tag{5.11}$$

where the asterisk superscript indicates an optimal quantity. To see that this control minimizes the Hamiltonian, check the second derivative of $\mathcal{H}$:

$$\frac{\partial^2 \mathcal{H}}{\partial u^2} = R^2 > 0 \; ;$$

hence, Equation (5.11) does minimize the Hamiltonian. Substituting Equation (5.11) into the state and adjoint equations, Equations (5.8) and (5.9), gives

$$\frac{d}{dP}\begin{bmatrix} z_1 \\ z_2 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ 2z_2/R - p_2/R^2 \\ 0 \\ 4z_2 - 2p_2/R - p_1 - (4 + R^2\omega^2)z_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2/R & 0 & -1/R^2 \\ 0 & 0 & 0 & 0 \\ 0 & -R^2\omega^2 & -1 & -2/R \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ p_1 \\ p_2 \end{bmatrix} . \tag{5.12}$$

Equation (5.12) is referred to as the Hamiltonian system associated with the optimal control problem. It is of the form

$$\frac{d}{dP}\begin{bmatrix} \vec{z} \\ \vec{p} \end{bmatrix} = \begin{bmatrix} W & -S \\ -Q & -W' \end{bmatrix} \begin{bmatrix} \vec{z} \\ \vec{p} \end{bmatrix} , \tag{5.13}$$

where

$$W = \begin{bmatrix} 0 & 1 \\ 0 & 2/R \end{bmatrix}, \; Q = \begin{bmatrix} 0 & 0 \\ 0 & R^2\omega^2 \end{bmatrix}, \text{ and } S = \begin{bmatrix} 0 & 0 \\ 0 & 1/R^2 \end{bmatrix} . \tag{5.14}$$

The solution to the linear homogeneous Equation (5.13) is

$$\begin{bmatrix} \vec{z} \\ \vec{p} \end{bmatrix}(P_1) = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}(P_1, P_2)\begin{bmatrix} \vec{z} \\ \vec{p} \end{bmatrix}(P_2)\ , \tag{5.15}$$

where the Ω_{ij} are (unknown) 2×2 matrices.

Two boundary conditions for Equation (5.15) come from the known initial conditions of the state variables. The other two boundary conditions for Equation (5.15) come from the transversality conditions. These pertain to the various requirements imposed on the final angle of approach. We consider two cases in the following sections. First, we consider the case where the final angle of approach is free. In this case the final angle of approach which minimizes the cost is automatically selected. Secondly, we consider a fixed final angle of approach. In this case the final angle of approach is selected by the user based on operational considerations. The trajectory is optimized subject to this constraint.

FREE FINAL ANGLE OF APPROACH - NECESSARY CONDITIONS

First, we assess the boundary conditions for the Hamiltonian system, Equation (5.12). The initial conditions on the state variables are known, which give the two conditions

$$\vec{z}(P = 0) = (z_{10}, z_{20})\ . \tag{5.16}$$

A free final angle of approach means that the terminal constraints on the state variables are free. According to the transversality conditions for a free end point,

$$\vec{p}(P = R_0) = 0\ . \tag{5.17}$$

(See Table 5-1 of Athans-Falb[7]). Integrating the third row of Equation (5.12) yields

$$p_1 = 0\ . \tag{5.18}$$

Substituting this into the second and fourth rows of Equation (5.12) gives

$$\frac{dz_2}{dP} = \frac{2z_2}{R} - \frac{p_2}{R^2} \tag{5.19}$$

and

$$\frac{dp_2}{dP} = -R^2\omega^2 z_2 - \frac{2}{R}p_2\ . \tag{5.20}$$

Differentiating Equation (5.20) gives

$$\begin{aligned}\frac{d^2p_2}{dP^2} &= -R^2\omega^2\frac{dz_2}{dP} - 2R\omega^2 z_2\frac{dR}{dP} - \frac{2}{R}\frac{dp_2}{dP} + \frac{2}{R^2}p_2\frac{dR}{dP} \\ &= -R^2\omega^2 \cdot (\frac{dz_2}{dP} - \frac{2z_2}{R}) - \frac{2}{R}\frac{dp_2}{dP} - \frac{2}{R^2}p_2\ .\end{aligned} \tag{5.21}$$

Applying Equation (5.19) and rearranging Equation (5.21) yields the following equation:

$$\frac{d^2p_2}{dP^2} + \frac{2}{R}\frac{dp_2}{dP} + (\frac{2}{R^2} - \omega^2)p_2 = 0\ . \tag{5.22}$$

The general solution to Equation (5.22) is

$$p_2(R) = C_1 R \sinh(\omega R) + C_2 R \cosh(\omega R)\ . \tag{5.23}$$

We can then solve for z_2 using Equation (5.20);

$$z_2 = -(\frac{2}{R}p_2 + \frac{dp_2}{dP})/(R^2\omega^2)$$

$$= -\frac{C_1}{R^2\omega^2}[\sinh(\omega R) - \omega R\cosh(\omega R)] - \frac{C_2}{R^2\omega^2}[\cosh(\omega R) - \omega R \sinh(\omega R)]\ . \tag{5.24}$$

To get the final solutions for p_2 and z_2 from Equations (5.23) and (5.24), we must evaluate C_1 and C_2 using the boundary constraints. The boundary condition given by Equation (5.17) does not help out. However, if the missile is to hit the target, the heading error angle at $R = 0$ must be zero ($\delta(0) = 0$). From physical considerations, $\delta(R) = x_2(R)$ is continuous, so the heading error angle should approach zero as R approaches zero; i.e.,

$$\lim_{R\to 0} x_2(R) = 0\ . \tag{5.25}$$

(Only continuous solutions for the state and adjoint variables are allowed.) When $C_1 = 0$ we have from Equation (5.24)

$$z_2(R) = -\frac{C_2}{R^2\omega^2}[\cosh(\omega R) - \omega R \sinh(\omega R)]$$

$$= -\frac{C_2}{R^2\omega^2}\left((1 + \frac{\omega^2}{2}R^2 + \frac{\omega^4}{24}R^4 + O(R^6)) - (\omega^2 R^2 + \frac{\omega^4}{6}R^4 + O(R^6))\right)$$

$$= -\frac{C_2}{R^2\omega^2}\cdot\left(1 - \frac{\omega^2}{2}R^2 - \frac{\omega^4}{8}R^4 + O(R^6)\right)\ . \tag{5.26}$$

Now, Equation (3.2) gives

$$x_2(R) = -\tan^{-1}(Rz_2) = -(Rz_2 - \frac{1}{3}(Rz_2)^3 + O((Rz_2)^5))$$

$$= -\frac{C_2}{8\cdot R}\frac{(-8 + 4(\omega R)^2 + (\omega R)^4)}{\omega^2} + \frac{C_2^3}{1536\cdot R^3}\frac{(-8 + 4(\omega R)^2 + (\omega R)^4)^3}{\omega^6} + \cdots$$

$$= -\frac{1}{3}\frac{C_2^3}{\omega^6}R^{-3} + \left(\frac{C_2}{\omega^2} + \frac{1}{2}\frac{C_2^3}{\omega^4}\right)R^{-1} + \left(-\frac{C_2}{2} - \frac{1}{8}\frac{C_2^3}{\omega^2}\right)R + \text{H.O.T.}$$

(H.O.T. stands for higher order terms.) This shows $x_2(R)$ goes to infinity as $R \to 0$ unless $C_2 = 0$. Hence, to satisfy Equation (5.25), it is necessary that $C_2 = 0$. When $C_2 = 0$ we have from Equation (5.24)

$$z_2(R) = -\frac{C_1}{R^2\omega^2}[\sinh(\omega R) - \omega R \cosh(\omega R)]$$

$$= -\frac{C_1}{R^2\omega^2}[(\omega R + \frac{\omega^3}{6}R^3 + O(R^5)) - (\omega R + \frac{\omega^3}{2}R^3 + O(R^5))]$$

$$= \frac{C_1}{3}\cdot \omega R + \text{H.O.T.}$$

$$x_2(R) = -\tan^{-1}(Rz_2) = -\frac{C_1}{3}\cdot\omega\cdot R^2 + \text{H.O.T.}$$

In this case Equation (5.25) is satisfied for all $0 < C_1 < \infty$. Note that the terms in the series expansion of x_2 associated with C_1 are even functions of R and the terms associated with C_2 are odd functions of R. This fact implies that we may do an analysis by taking $C_1 = 0$ and $C_2 \neq 0$ and then $C_1 \neq 0$ and $C_2 = 0$. (This was taken into account in the analysis presented in this paragraph.)

With $C_2 = 0$, we get

$$z_2(R) = -\frac{C_1}{R^2\omega^2}[\sinh(\omega R) - \omega R\cosh(\omega R)] \tag{5.27}$$

and

$$p_2(R) = C_1 R\sinh(\omega R)\ . \tag{5.28}$$

From the first row of Equation (5.12) and Equation (5.27)

$$z_1(R) = -C_1\frac{\sinh(\omega R)}{R\omega^2} + K_2\ . \tag{5.29}$$

The integration constants C_1 and K_2 may be computed from the initial values of γ and δ. From Equations (3.1), (5.16) and (5.27) we have

$$-\frac{\tan(\delta(R_0))}{R_0} = -\frac{C_1}{R_0^2\omega^2}[\sinh(\omega R_0) - \omega R_0\cosh(\omega R_0)]\ .$$

Solving this equation in C_1 gives

$$C_1 = \frac{R_0\omega^2\tan(\delta(R_0))}{\sinh(\omega R_0) - \omega R_0\cosh(\omega R_0)}\ . \tag{5.30}$$

From Equations (3.1), (5.16) and (5.29) we have

$$\gamma_0 - \delta_0 = -C_1\frac{\sinh(\omega R_0)}{R_0\omega^2} + K_2\ .$$

Hence,

$$K_2 = \frac{C_1\sinh(\omega R_0)}{R_0^2\omega^2} + \gamma_0 - \delta_0\ . \tag{5.31}$$

Next, closed form solutions for the (untransformed) state variables are given. From Equations (3.2), (5.27) and (5.29) we get

$$\gamma(R) = -C_1\frac{\sinh(\omega R)}{R\omega^2} + K_2 + \tan^{-1}(\frac{C_1}{R\omega^2}[\sinh(\omega R) - \omega R\cosh(\omega R)])\ , \tag{5.32}$$

and

$$\delta(R) = \tan^{-1}(\frac{C_1}{R\omega^2}[\sinh(\omega R) - \omega R \cosh(\omega R)]) \ . \tag{5.33}$$

Such a closed form solution for the state variables given by Equations (5.32), (5.33), (5.30) and (5.31) was not obtained in the original solution method of this problem. It is reiterated that the closed form solution obtained here is an approximated solution to the optimal control problem given by Equations (2.1) and (2.2) because of the quadratic approximation made in the cost function in Equation (4.4). (No other approximation is made.)

FREE FINAL ANGLE OF APPROACH - SUFFICIENT CONDITIONS

In this section sufficient conditions for the optimal control problem are considered. The sufficient conditions involve solving the Hamilton-Jacobi partial differential equation. A closed form solution to the Hamilton-Jacobi equation is found for the free final angle of approach case. This solution also gives a closed form solution to the cost. We will see that the necessary conditions presented previously are sufficient for optimality.

There is a relationship between the state and adjoint variables. We demonstrate this next, which will also develop the Riccati equation. Setting $P_1 = R_0$ and $P_2 = P$ in Equation (5.15) and using Equations (5.17) and (5.18) we get

$$0 = \Omega_{21}(R_0, P)\vec{z}(P) + \Omega_{22}(R_0, P)\vec{p}(P)$$

so that

$$\vec{p}(P) = -\Omega_{22}(R_0, P)^{-1}\Omega_{21}(R_0, P)\vec{z}(P) \triangleq K(P)\vec{z}(P) \ . \tag{5.34}$$

Substituting Equation (5.34) into Equation (5.13), we get

$$\frac{d}{dP}\begin{bmatrix} \vec{z} \\ \vec{p} \end{bmatrix} = \begin{bmatrix} (W - SK)\vec{z} \\ (-Q - W'K)\vec{z} \end{bmatrix} .$$

From this and differentiating Equation (5.34),

$$(-Q - W'K)\vec{z} = (KW - KSK + \frac{dK}{dP})\vec{z} \ . \tag{5.35}$$

Equation (5.35) holds for all $\vec{z}$; hence,

$$\frac{dK}{dP} + KW + W'K - KSK + Q = 0 \ . \tag{5.36}$$

Equation (5.36) is the Riccati differential equation for the LQR optimal control problem. It may be shown that $K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$ is symmetric; that is $k_{12} = k_{21}$, and is positive definite, see Athans-Falb.[7] From Equations (5.18), (5.27), (5.28) and (5.29), we see that

$$\begin{bmatrix} 0 \\ p_2(P) \end{bmatrix} = \begin{bmatrix} k_{11}(P) & k_{12}(P) \\ k_{12}(P) & k_{22}(P) \end{bmatrix} \begin{bmatrix} z_1(P) \\ z_2(P) \end{bmatrix} = \begin{bmatrix} k_{11}(P)z_1(P) + k_{12}(P)z_2(P) \\ k_{12}(P)z_1(P) + k_{22}(P)z_2(P) \end{bmatrix} , \tag{5.37}$$

where the k'^s_{ij} are the unknowns. Equation (5.37) is overdetermined; hence, we select $k_{11}$to be a free parameter. Solving the first row of Equation (5.37) for k_{12} gives

$$k_{12} = -\frac{k_{11}z_1}{z_2} \,. \tag{5.38}$$

Substituting Equation (5.38) into the second row of Equation (5.37) gives

$$k_{22} = \frac{1}{z_2}(p_2 + \frac{k_{11}z_1^2}{z_2}) \,. \tag{5.39}$$

Hence,

$$K(P) = \begin{bmatrix} k_{11}(P) & -\frac{k_{11}(P)z_1(P)}{z_2(P)} \\ -\frac{k_{11}(P)z_1(P)}{z_2(P)} & \frac{1}{z_2(P)} \cdot \left(p_2(P) + \frac{k_{11}(P)z_1^2(P)}{z_2(P)}\right) \end{bmatrix} \,. \tag{5.40}$$

Equation (5.36) has a terminal (not initial) condition, which is $K(R_0) = 0$ (see Table 5-1 of Athans-Falb[7]). A solution to Equation (5.36) which satisfies this terminal condition is unique (see Athans-Falb,[7] page 762). It may be seen that

$$K(R) = \begin{bmatrix} 0 & 0 \\ 0 & -\left(\frac{R^3\omega^2\sinh(\omega R)}{\sinh(\omega R) - \omega R\cosh(\omega R)}\right) \end{bmatrix} \tag{5.41}$$

fits. The solution presented by Equation (5.41) was obtained by setting $k_{11} = 0$ in Equation (5.40).

Sufficient conditions for the optimality are considered next. This involves showing that the feedback control given by Equation (5.11) satisfies these conditions, and it will be seen that it is globally optimal. We refer to Theorem 5-13 of Athans-Falb.[7] First, we see from Equation (5.7) that the Hamiltonian is normal relative to $X = \Re^2 \times (0, R_0)$. From the work done previously in this section, it is seen that this controller is the $\mathcal{H}$-minimal control relative to X.

We will find it convenient to rewrite the Hamiltonian. Consider the Hamiltonian as given by Equation (5.7). It may be rewritten as

$$\mathcal{H} = \frac{1}{2}\begin{bmatrix} \vec{z}' & u \end{bmatrix}\begin{bmatrix} L_{11} & L_{12} \\ L'_{12} & L_{22} \end{bmatrix}\begin{bmatrix} \vec{z} \\ u \end{bmatrix} + \left\langle \vec{p}, A\vec{z} + Bu \right\rangle \tag{5.42}$$

where

$$L_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 4 + R^2\omega^2 \end{bmatrix}, \; L_{12} = \begin{bmatrix} 0 \\ -2R \end{bmatrix} \text{ and } L_{22} = R^2 \,. \tag{5.43}$$

Comparing Equation (5.14) with Equations (5.2) and (5.43), we find that

$$W = A - \frac{BL'_{12}}{L_{22}}, \; S = \frac{BB'}{L_{22}}, \; Q = L_{11} - \frac{L_{12}L'_{12}}{L_{22}} \,.$$

Substituting these into the Riccati equation, Equation (5.36), gives

$$\frac{dK}{dP}+KA-K\frac{BL_{12}'}{L_{22}}+A'K-\frac{L_{12}B'}{L_{22}}K-K\frac{BB'}{L_{22}}K+L_{11}-\frac{L_{12}L_{12}'}{L_{22}}=0\ . \tag{5.44}$$

The last part of the sufficient conditions in Theorem 5-13 of Athans-Falb[7] is the Hamilton-Jacobi partial differential equation. The Hamilton-Jacobi partial differential equation is

$$\frac{\partial \mathcal{J}}{\partial P}+\min_u \mathcal{H}(\vec{z},\frac{\partial \mathcal{J}}{\partial \vec{z}},u,P)=0\ . \tag{5.45}$$

Let $\mathcal{J}=\frac{1}{2}\left\langle \vec{z},K\vec{z}\right\rangle$. Then $\frac{\partial \mathcal{J}}{\partial P}=\frac{1}{2}\left\langle \vec{z},\frac{dK}{dP}\vec{z}\right\rangle$ and $\frac{\partial \mathcal{J}}{\partial \vec{z}}=K\vec{z}$. From Equations (5.11) and (5.43), we have that $u^*=-\left[\frac{L_{12}'}{L_{22}}+\frac{B'K}{L_{22}}\right]\vec{z}$. Substituting into the Hamilton-Jacobi equation,

$$\begin{aligned}
&\frac{1}{2}\left\langle \vec{z},\frac{dK}{dP}\vec{z}\right\rangle+\min_u\left\{\frac{1}{2}\left\langle \vec{z},L_{11}\vec{z}\right\rangle+\left\langle \vec{z},L_{12}u\right\rangle+\frac{1}{2}\left\langle u,L_{22}u\right\rangle+\left\langle K\vec{z},A\vec{z}+Bu\right\rangle\right\}\\
&=\frac{1}{2}\left\langle \vec{z},\frac{dK}{dP}\vec{z}\right\rangle+\frac{1}{2}\left\langle \vec{z},L_{11}\vec{z}\right\rangle-\frac{1}{L_{22}}\left\langle \vec{z},L_{12}(L_{12}'+B'K)\vec{z}\right\rangle\\
&+\frac{1}{2\cdot L_{22}}\left\langle (L_{12}'+B'K)\vec{z},(L_{12}'+B'K)\vec{z}\right\rangle+\frac{1}{2}\left\langle \vec{z},A'K\vec{z}\right\rangle+\frac{1}{2}\left\langle \vec{z},KA\vec{z}\right\rangle\\
&-\frac{1}{L_{22}}\left\langle \vec{z},KB(L_{12}'+B'K)\vec{z}\right\rangle\\
&=\frac{1}{2}\left\langle \vec{z},\frac{dK}{dP}\vec{z}\right\rangle+\frac{1}{2}\left\langle \vec{z},L_{11}\vec{z}\right\rangle+\frac{1}{2}\left\langle \vec{z},A'K\vec{z}\right\rangle+\frac{1}{2}\left\langle \vec{z},KA\vec{z}\right\rangle\\
&-\frac{1}{2\cdot L_{22}}\left\langle \vec{z},KBL_{12}'\vec{z}\right\rangle-\frac{1}{2\cdot L_{22}}\left\langle \vec{z},KBB'K\vec{z}\right\rangle\\
&-\frac{1}{2\cdot L_{22}}\left\langle \vec{z},L_{12}B'K\vec{z}\right\rangle-\frac{1}{2\cdot L_{22}}\left\langle \vec{z},L_{12}L_{12}'\vec{z}\right\rangle\ .
\end{aligned} \tag{5.46}$$

Since Equation (5.46) holds for all values of $\vec{z}$, we get that

$$\begin{aligned}
&\frac{dK}{dP}+L_{11}+A'K+KA-\frac{1}{L_{22}}\cdot KBL_{12}'-\frac{1}{L_{22}}\cdot KBB'K\\
&-\frac{1}{L_{22}}\cdot L_{12}B'K-\frac{1}{L_{22}}\cdot L_{12}L_{12}'=0\ ,
\end{aligned}$$

and the Hamilton-Jacobi partial differential equation is satisfied since K satisfies the Riccati equation. As a consequence of this, we get a closed form solution for the optimal cost ,

$$\mathcal{J}(\gamma,\delta,P,\kappa)=\frac{1}{2}\left\langle F(\vec{x},P),K(P)F(\vec{x},P)\right\rangle+\int_0^{R_0}\omega^2dP\ , \tag{5.47}$$

where F is the transformation given by Equation (3.1). Equation (5.47) will be verified in Chapter 6, the Numerical Examples chapter. The last term in Equation (5.47) is to account for the "$+\omega^2$" term that was deleted in obtaining the quadratic cost approximation in Chapter 4.

TERMINAL CONSTRAINT ON INTERCEPT ANGLE - NECESSARY CONDITIONS

In this section, the LQR optimal control problem will be formulated with a terminal constraint on the intercept angle. The state variable x_1 represents the velocity vector angle, which is the intercept angle. The terminal constraint then is $x_1(P = R_0) = \gamma_f$. In the transformed coordinates, see Equation (3.1), using the continuity of the state variables, the terminal constraint is

$$\lim_{R\to 0} \left[z_1 - \tan^{-1}(Rz_2)\right] = \gamma_f \ . \tag{5.48}$$

The two initial boundary conditions for Equation (5.12) are

$$\vec{z}(P = 0) = (z_{10}, z_{20}) \ . \tag{5.49}$$

An integration of Equation (5.12c) gives

$$p_1 = constant = K_1 \ , \tag{5.50}$$

where K_1 is to be determined. (Integration constants in this section are not to be confused with integration constants of the previous section.) The remaining state and adjoint equations are

$$\frac{dz_1}{dP} = z_2 \ , \tag{5.51}$$

$$\frac{dz_2}{dP} = 2\frac{z_2}{R} - \frac{p_2}{R^2} \ , \tag{5.52}$$

and

$$\frac{dp_2}{dP} = -K_1 - R^2\omega^2 z_2 - \frac{2p_2}{R} \ . \tag{5.53}$$

The same differential equation as Equation (5.21) can be derived for p_2 with the same general solution:

$$p_2(R) = C_1 R\sinh(\omega R) + C_2 R\cosh(\omega R) \ . \tag{5.54}$$

Applying Equations (5.53) and (5.54),

$$z_2(R) = -(2\frac{p_2}{R} + \frac{dp_2}{dP} + K_1)/(R^2\omega^2)$$

$$= -\frac{C_1}{R^2\omega^2}[\sinh(\omega R) - \omega R\cosh(\omega R)] - \frac{C_2}{R^2\omega^2}[\cosh(\omega R) - \omega R\sinh(\omega R)] + \frac{K_1}{R^2\omega^2} \ . \tag{5.55}$$

The same situation as before applies and is used to evaluate the coefficients. Considering the Taylor series expansion of x_2, using Equations (5.55) and (3.1), the term $K_1/R^2\omega^2$ will contribute terms with even powers of R. Referring to Equation (5.26), the Taylor series expansion of z_2 when $C_1 = 0$ is

$$z_2(R) = -\frac{C_2}{R^2\omega^2} \cdot \left(1 - \frac{\omega^2}{2}R^2 - \frac{\omega^4}{8}R^4 + O(R^6)\right) + \frac{K_1}{R^2\omega^2} \ .$$

From this, it can be seen that a requirement for satisfying Equation (5.25) is that $K_1 = C_2$. Therefore,

$$z_2(R) = -\frac{C_1}{R^2\omega^2}[\sinh(\omega R) - \omega R\cosh(\omega R)] - \frac{C_2}{R^2\omega^2}[\cosh(\omega R) - \omega R\sinh(\omega R) - 1] \; . \tag{5.56}$$

From the first row of Equation (5.12) and Equation (5.56) we get

$$z_1(R) = -\frac{1}{R\omega^2}[C_1\sinh(\omega R) + C_2\cosh(\omega R) - C_2] + K_2 \; , \tag{5.57}$$

where K_2 is an integration constant. Applying the terminal constraint, Equation (5.48), we get

$$K_2 = \frac{C_1}{\omega} + \gamma_f \; . \tag{5.58}$$

Hence,

$$z_1(R) = -\frac{1}{R\omega^2}[C_1\sinh(\omega R) + C_2\cosh(\omega R) - C_2] + \frac{C_1}{\omega} + \gamma_f \; . \tag{5.59}$$

The constants C_1 and C_2 can be determined from the initial state $\vec{x}(R_0)$. From Equations (5.55) and (5.59) we obtain

$$z_1(R_0) = \gamma_0 - \delta_0 = -\frac{1}{R_0\omega^2}[C_1\sinh(\omega R_0) + C_2\cosh(\omega R_0) - C_2] + \frac{C_1}{\omega} + \gamma_f \; ,$$

and

$$z_2(R_0) = -\tan(\delta_0)/R_0$$

$$= -\frac{C_1}{R_0^2\omega^2}[\sinh(\omega R_0) - \omega R_0\cosh(\omega R_0)] - \frac{C_2}{R_0^2\omega^2}[\cosh(\omega R_0) - \omega R_0\sinh(\omega R_0) - 1] \; .$$

Hence, C_1 and C_2 are the solution of the linear algebraic equation:

$$\begin{bmatrix} R_0\omega - \sinh(\omega R_0) & 1 - \cosh(\omega R_0) \\ \begin{pmatrix} -\sinh(\omega R_0) \\ +\omega R_0\cosh(\omega R_0) \end{pmatrix} & \begin{pmatrix} -\cosh(\omega R_0) + 1 \\ +\omega R_0\sinh(\omega R_0) \end{pmatrix} \end{bmatrix}$$

$$\cdot \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = R_0\omega^2 \begin{bmatrix} \gamma_0 - \delta_0 - \gamma_f \\ -\tan(\delta_0) \end{bmatrix} \; . \tag{5.60}$$

At this point, the closed form solutions to the (untransformed) state variables for the constrained final angle of approach case may be obtained. We have

$$\gamma(R) = -\frac{1}{R\omega^2}[C_1\sinh(\omega R) + C_2\cosh(\omega R) - C_2] + K_2$$

$$+\tan^{-1}\left(\frac{C_1}{R\omega^2}[\sinh(\omega R) - \omega R\cosh(\omega R)] + \frac{C_2}{R\omega^2}[\cosh(\omega R) - \omega R\sinh(\omega R) - 1]\right) \; , \tag{5.61}$$

and

$$\delta(R) = \tan^{-1}\left(\frac{C_1}{R\omega^2}[\sinh(\omega R) - \omega R\cosh(\omega R)] + \frac{C_2}{R\omega^2}[\cosh(\omega R) - \omega R\sinh(\omega R) - 1]\right) \; . \tag{5.62}$$

We have arrived at closed form solutions for the state variables for the constrained intercept angle case. These are given by Equations (5.61) and (5.62). Such closed form solutions were not obtained in the original formulation. See Lin.[1]

RELATIONSHIP BETWEEN STATE AND ADJOINT VARIABLES - TERMINAL CONSTRAINT ON INTERCEPT ANGLE CASE

We first formulate the Riccati equation for this case, which also gives a relationship between the state and adjoint variables. Setting $P_1 = 0$ and $P_2 = P$ in the first row of Equation (5.15) we get

$$\vec{z}(P) = -\Omega_{11}^{-1}(0,P)\Omega_{12}(0,P)\vec{p}(P) + \Omega_{11}^{-1}(0,P)\vec{z}(0) \,. \tag{5.63}$$

Set $M(P) = \Omega_{11}^{-1}(0,P) \cdot \Omega_{12}(0,P)$. Differentiating Equation (5.63) gives

$$\frac{d\vec{z}}{dP} = \frac{dM}{dP}\vec{p} + M\frac{d\vec{p}}{dP} + \frac{d\Omega_{11}^{-1}}{dP}\vec{z}(0) \,. \tag{5.64}$$

Substituting Equation (5.63) into the second row of Equation (5.13) yields

$$\frac{d\vec{p}}{dP} = -Q[M\vec{p} + \Omega_{11}^{-1}(0,P)\vec{z}(0)] + W'\vec{p} \,. \tag{5.65}$$

Substituting Equation (5.65) into Equation (5.64) gives

$$\frac{d\vec{z}}{dP} = \frac{dM}{dP}\vec{p} - MQM\vec{p} - MW'\vec{p} - MQ\Omega_{11}^{-1}(0,P)\vec{z}(0) + \frac{d\Omega_{11}^{-1}}{dP}\vec{z}(0) \,. \tag{5.66}$$

Substituting Equation (5.63) into the first row of Equation (5.13) gives

$$\frac{d\vec{z}}{dP} = WM\vec{p} + W\Omega_{11}^{-1}(0,P)\vec{z}(0) - S\vec{p} \,. \tag{5.67}$$

Equating Equations (5.66) and (5.67), since these equations hold for any $\vec{p}$ and $\vec{z}(0)$, gives the following two equations :

$$\frac{dM}{dP} - MQM - MW' - WM + S = 0 \tag{5.68}$$

and

$$\frac{d\Omega_{11}^{-1}}{dP} - W\Omega_{11}^{-1}(0,P) - MQ\Omega_{11}^{-1}(0,P) = 0 \,. \tag{5.69}$$

The boundary constraint for Equation (5.68) is

$$M(0) = -\Omega_{11}^{-1}(0,0)\Omega_{12}(0,0) = I \cdot 0 = 0 \,. \tag{5.70}$$

The initial condition for Equation (5.69) is $\Omega_{11}^{-1}(0,0) = I$. Equations (5.63), (5.68), (5.69) and (5.70) give the relationship between the state variables $\vec{z}$ and the adjoint variables $\vec{p}$ in this case.

CHAPTER 6

NUMERICAL EXAMPLES

In this chapter, several numerical examples of the linearized κ guidance are given. These numerical examples are an attempt to evaluate the performance of the linearized kappa guidance method. A comparison to the original method, which is presented in Lin[1], is given. First, individual trajectories generated by each method are compared. These results are shown in Figures 6-1 through 6-4. Each figure shows a trajectory generated by the original κ method and a trajectory generated by the linearized κ method. In each case, the origin of the trajectory (launch point of the missile) is at $(0,0)$ and the PIP is at $(1,0)$, giving $R_0 = 1$. The constant ω is set to one. For all of the these examples, the program was set to stop when $R \leq 0.01$. The abscissa on these figures is downrange and the ordinate is altitude above a reference launch altitude h_{ref} (plots depict range vs. h_{ref}.) No particular units are represented; however, these figures may be considered to be in miles. In Figure 6-1, $\gamma_0 = \delta_0 = \pi/3$ and $\gamma_f =$ FREE. In Figures 6-2, 6-3 and 6-4, $\gamma_0 = \delta_0 = \pi/3$. In Figure 6-2, $\gamma_f = -\pi/2$. In Figure 6-3, $\gamma_f = 0$. In Figure 6-4, $\gamma_f = \pi/2$. The trajectory in Figure 6-4 is presented for the purpose of demonstrating a terminal approach opposite that of Figure 6-2 and to provide a more complete comparison of the linearized κ guidance method with the original κ guidance method.

Figures 6-5 through 6-7 are cost curves. Recall that smaller cost means a higher intercept velocity. Each of these figures shows the costs for a particular family of trajectories. For each trajectory whose cost is shown in Figures 6-5 through 6-7, the origin of the trajectory is at $(0,0)$ and the PIP is at $(1,0)$. Equation (2.2) is used to compute the cost of every trajectory in this report. In Figure 6-5, the costs of trajectories with free final angle of approach with various initial heading error angles are computed. Each point of the abscissa represents one of these trajectories with a specific initial heading error angle. The corrseponding ordinate value is the cost associated with that angle. Trajectories with initial heading error angles within $\delta_0 \in [-1.5, 1.5]$ are computed. For the linearized method, it was not possible to compute trajectories for $|\delta_0| > 1.2$. It may be seen that the costs of trajectories computed with the linearized method are close to those computed with the original method for $|\delta_0| < 0.8$. The third curve shown in Figure 6-5 gives costs for various δ_0 using Equation (5.47). It is seen that Equation (5.47) is a close approximation to the true cost of a trajectory, computed by either method, for $\delta_0 \in [-0.5, 0.5]$.

In Figures 6-6 and 6-7, the initial heading error angle is fixed and the final angle of approach (or final velocity vector angle) is varied. In Figure 6-6 we have $\delta_0 = 0$, and in Figure 6-7 we have $\delta_0 = \pi/3$. That is, for Figure 6-6, costs for trajectories, computed with both methods with $\delta_0 = 0$, are plotted for $\gamma_f \in [-1.5, 1.5]$. The costs in this figure computed by each method, for a γ_f are nearly identical. The costs for trajectories computed by the two methods in Figure 6-7 are still comparable, although they are not as close as in Figure 6-6. The linearized method has a slightly lower cost for $\gamma_f \in [-1.5, -1.1]$, while the original method has the lower cost for $\gamma_f \in [-1.1, 1, 5]$.

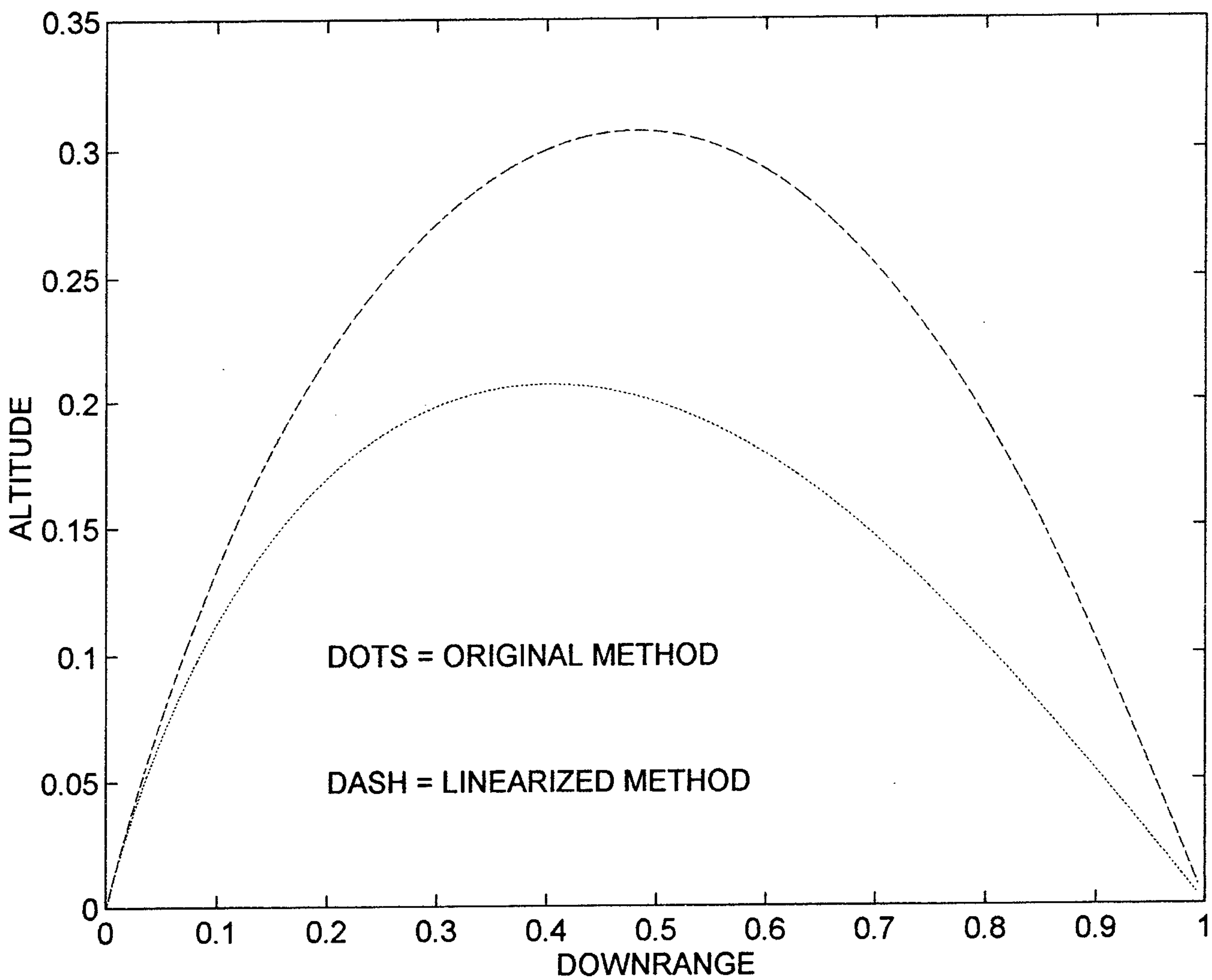

FIGURE 6-1. $\gamma_0 = \delta_0 = \pi/3, \gamma_f = \text{FREE}$

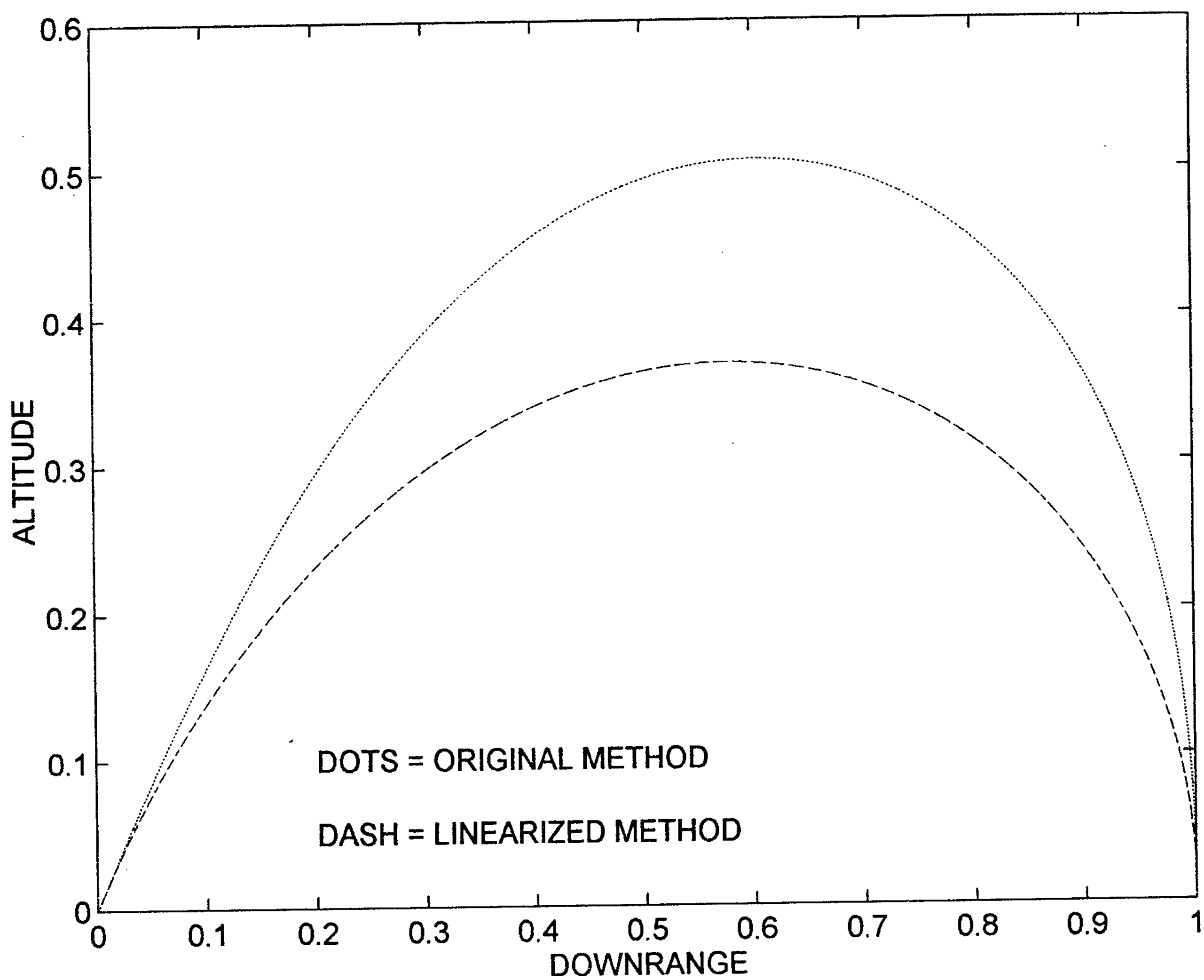

FIGURE 6-2. $\gamma_0 = \delta_0 = \pi/3, \gamma_f = -\pi/2$

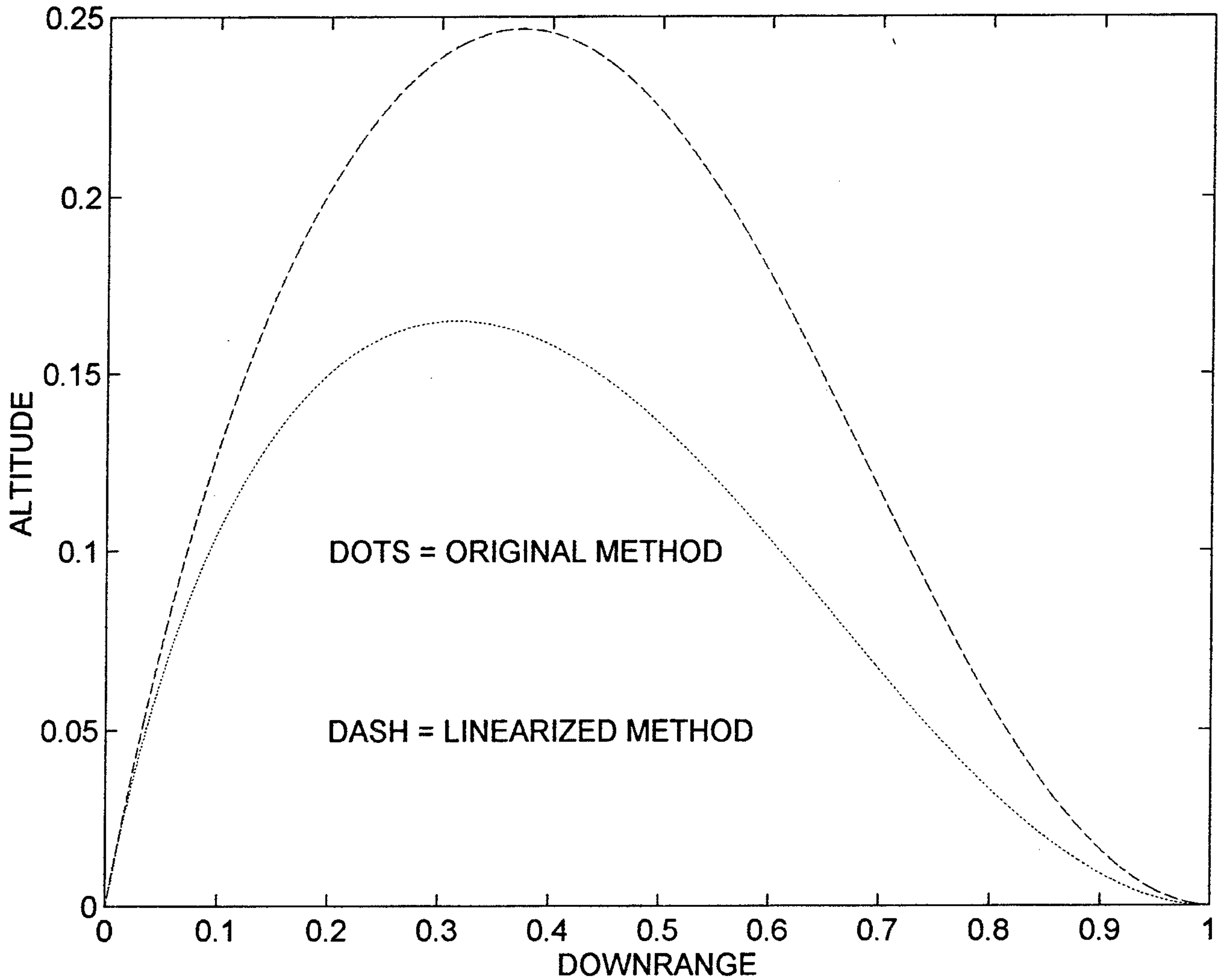

FIGURE 6-3. $\gamma_0 = \delta_0 = \pi/3, \gamma_f = 0$

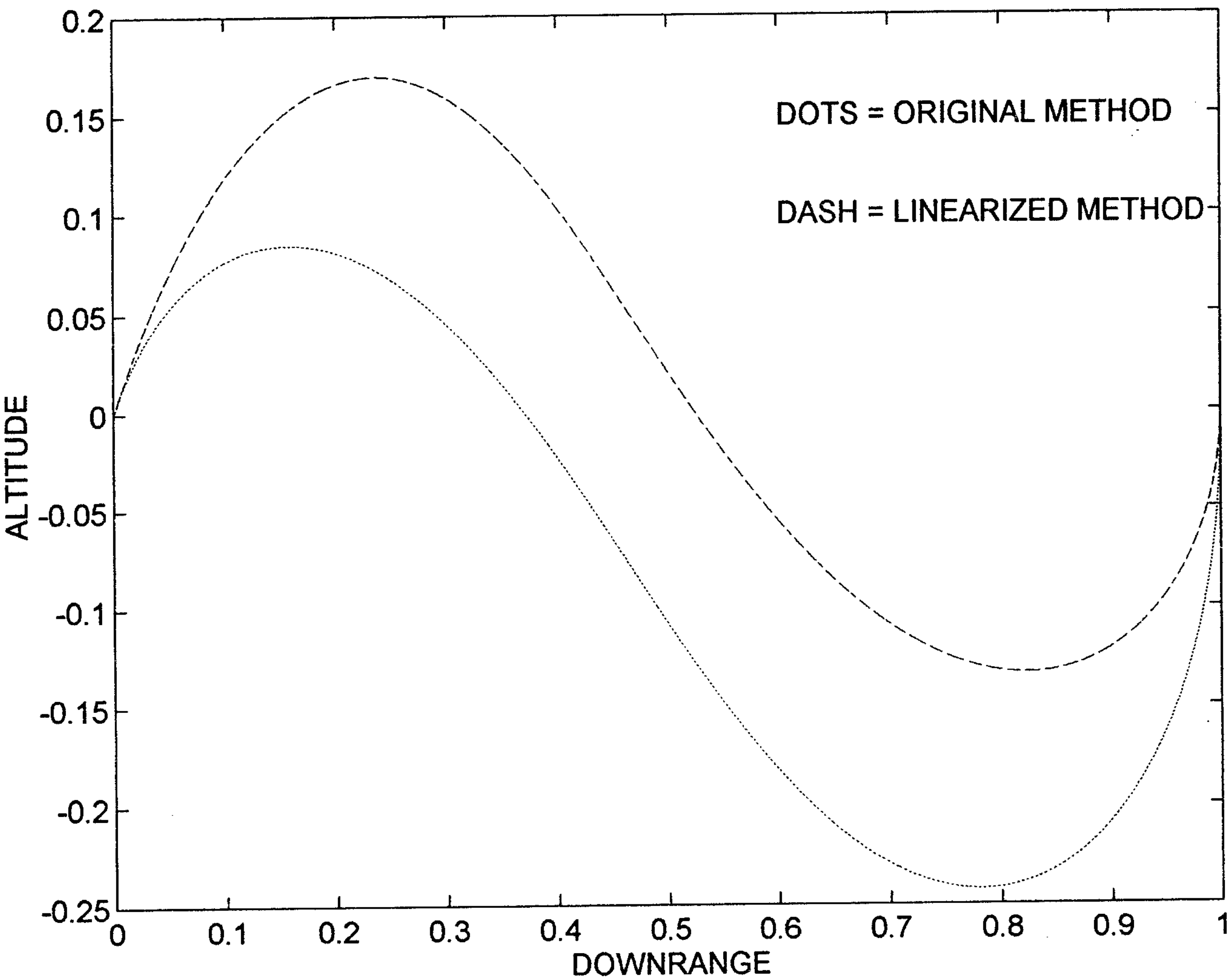

FIGURE 6-4. $\gamma_0 = \delta_0 = \pi/3, \gamma_f = \pi/2$

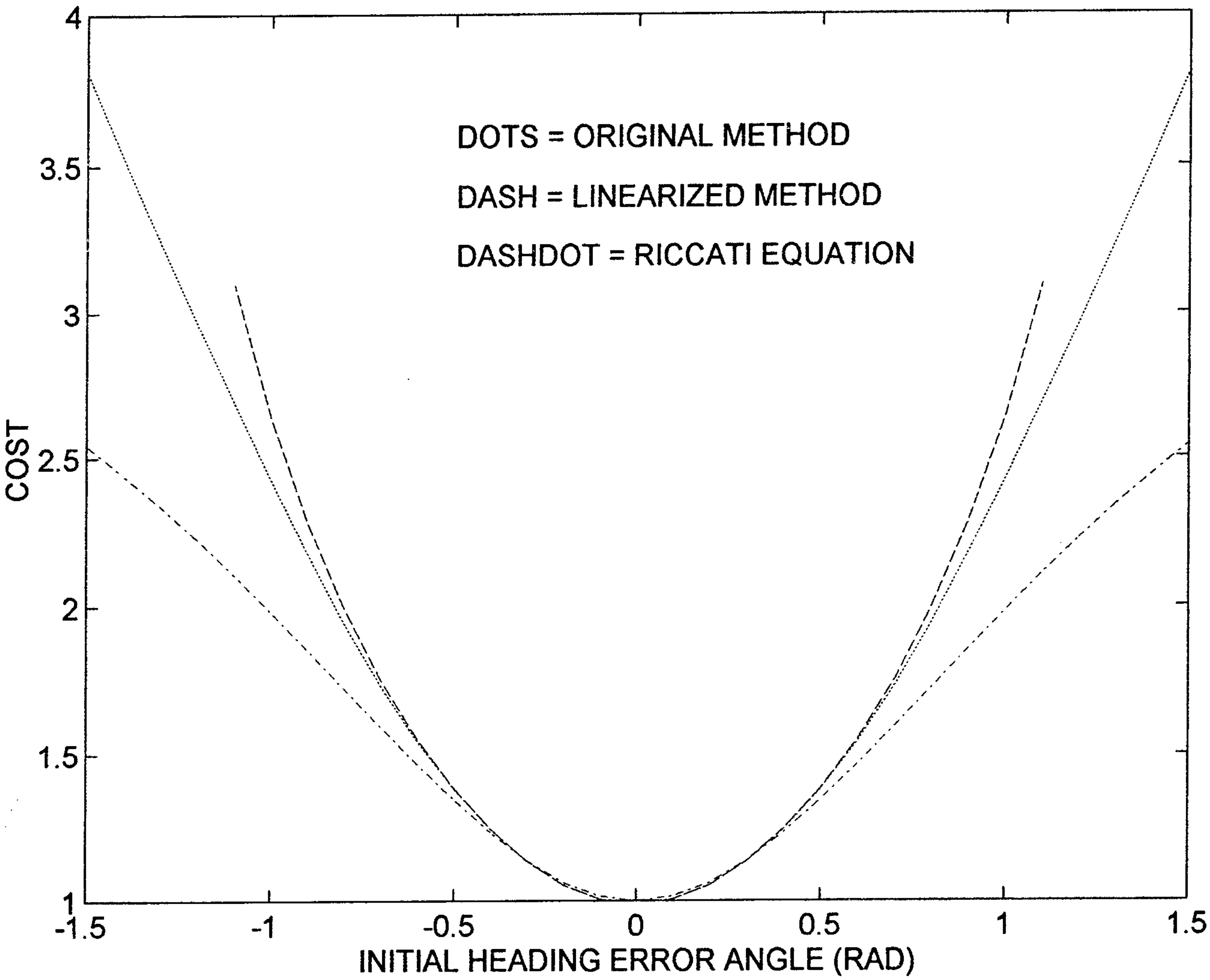

FIGURE 6-5. COST CURVE FOR $\gamma_0 = \delta_0 \in [-1.5, 1.5], \gamma_f = $ FREE

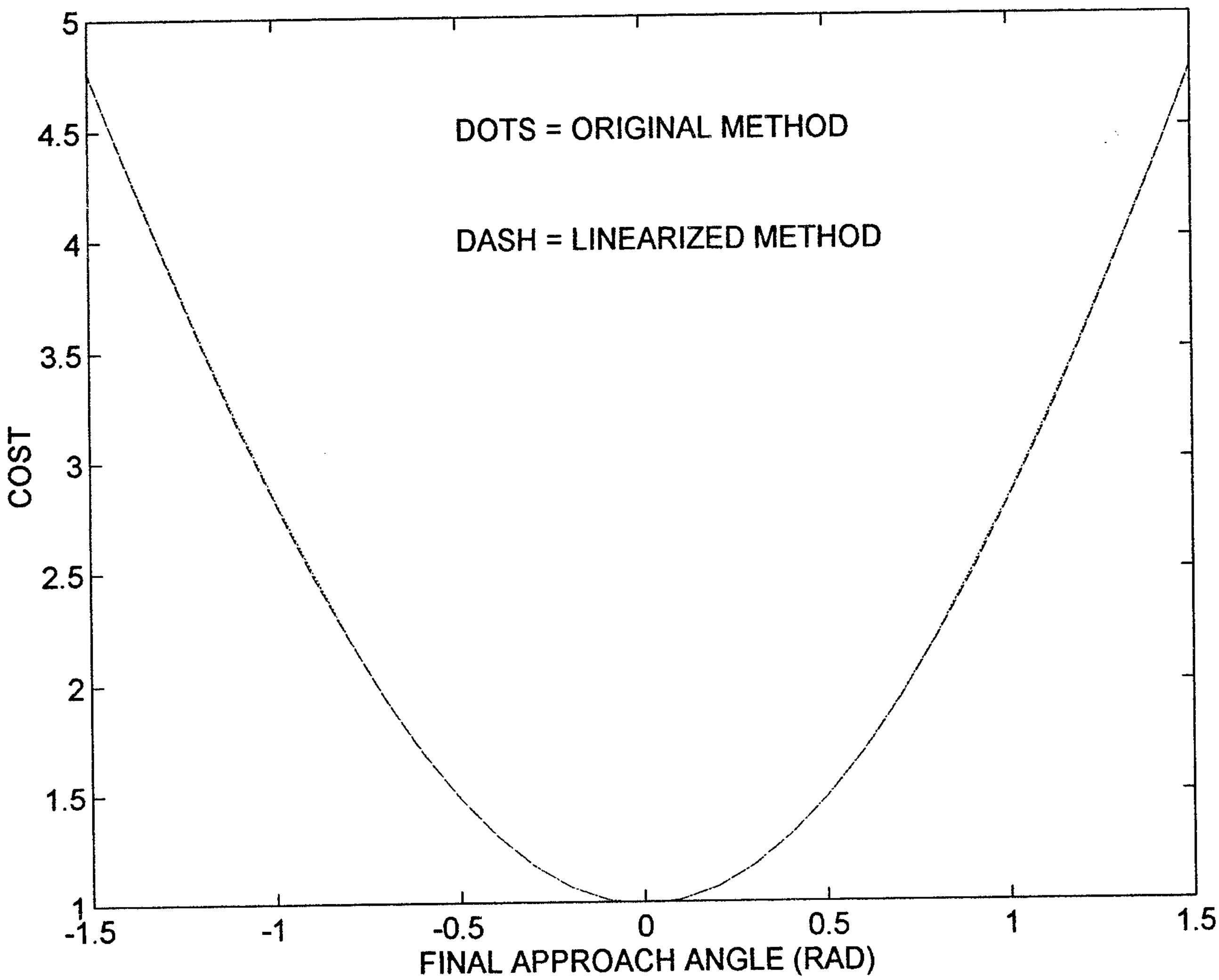

FIGURE 6-6. COST CURVE FOR $\gamma_0 = \delta_0 = 0, \gamma_f \in [-1.5, 1.5]$

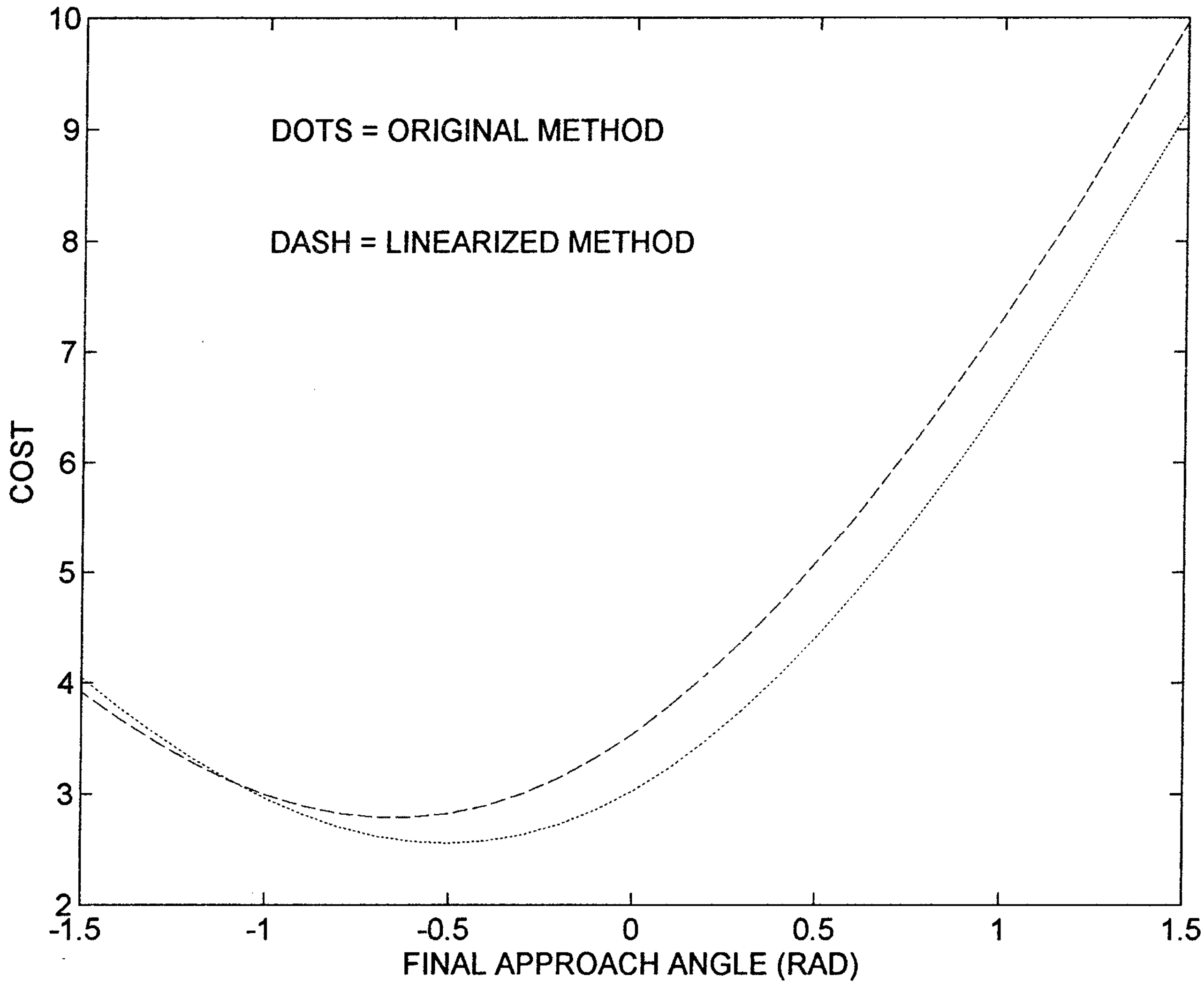

FIGURE 6-7. COST CURVE FOR $\gamma_0 = \delta_0 = \pi/3, \gamma_f \in [-1.5, 1.5]$

Figures 6-8 and 6-9 are cost trajectories. These figures show the accumulating cost of a trajectory from the origin to the PIP. For a general system with n dimensional state space, a trajectory is plotted in $(n+2)$ dimensional space; see Athans-Falb,[7] Figure 5-17(b), page 312. Projecting onto the cost-time (the independent parameter) plane gives curves such as in Figures 6-8 and 6-9. In Figure 6-8 we have $\delta_0 = \pi/3$, and the final angle of approach is free. Cost trajectories for both methods are plotted. The final costs for these cost trajectories are given in Figure 6-5. In Figure 6-9 we have $\delta_0 = \pi/3$ and the fixed final angle of approach is $\gamma_f = -\pi/2$. The costs of these trajectories are given in Figure 6-7.

Figures 6-10, 6-11 and 6-12 are cost curves as Figures 6-5, 6-6 and 6-7. The same cost curves for the original method are plotted as in Figure 6-5, 6-6 and 6-7, that is, in Figure 6-10 we have $\delta_0 \in [-1.5, 1.5]$ and γ_f is free, in Figure 6-11 we have $\delta_0 = 0$ and $\gamma_f \in [-1.5, 1.5]$ and in Figure 6-12 we have $\delta_0 = \pi/3$ and $\gamma_f \in [-1.5, 1.5]$. Instead of comparing with the linearized method, as was done in Figures 6-5, 6-6 and 6-7, a parabola was fitted over the curve in each of the figures. The parabolic curve fit is close in each of these figures. This close fit indicates that the cost has the potential of being functionalized. A functionalized cost curve may be useful in operational situations. For example, Figure 6-11 shows that it would be straightforward to compute cost for intercepting at different angles. Cost could be computed for all angles by computing for two angles since two angles would determine a parabola (with one angle at the minimum).

As was mentioned previously, both the original method and the linearized method give suboptimal solutions that make different approximations. Since these two methods generate two different suboptimal solutions, we do not expect that their corresponding trajectories match up. This is why the trajectories in Figures 6-1 through 6-4 do not match. Looking at the trajectories in Figure 6-1 for example, the trajectory by the linearized method is longer and attains a higher altitude than the trajectory by the original method. It seems that the original method gives a trajectory with a higher curvature early on which results in a fast reduction in the heading error angle. The trajectory generated by the linearized method has a smaller curvature early in the trajectory which results in a gradual reduction in the heading error angle. The cost function, Equation (2.2), involves both of these factors (curvature and heading error angle) and evidently heading error angle can only be reduced through the use of curvature. One may suggest that the flight times (from launch to arrival at the PIP) of a missile actually flying the two trajectories may not be the same. For the original method, the higher curvature at the beginning will reduce speed to a greater extent; however, the path is shorter. For the linearized method the lower curvature will allow a more gradual reduction in speed. Any time difference, if it does exist, is not relevant since the cost criterion for the purposes of this report is given by Equation (2.2), which (as stated previously) maximizes terminal velocity.

The trajectories in these figures were computed differently for each method. Considering the linearized κ guidance method, first an incremental length along the trajectory, $\triangle s$, is set. The current values of the state variables, the downrange and altitude coordinates, and the range R to the PIP are assumed given. These variables are substituted into Equations (5.32) and (5.33) for the free terminal constraint case and into Equations (5.61) and (5.62) for the fixed final angle of approach case. These formulas give the new velocity vector angle, γ, the missile should have. Using γ and $\triangle s$, the trajectory is incremented. What has been described is an Euler method, while what is actually used is a two-point method. The velocity vector angle used is the average of

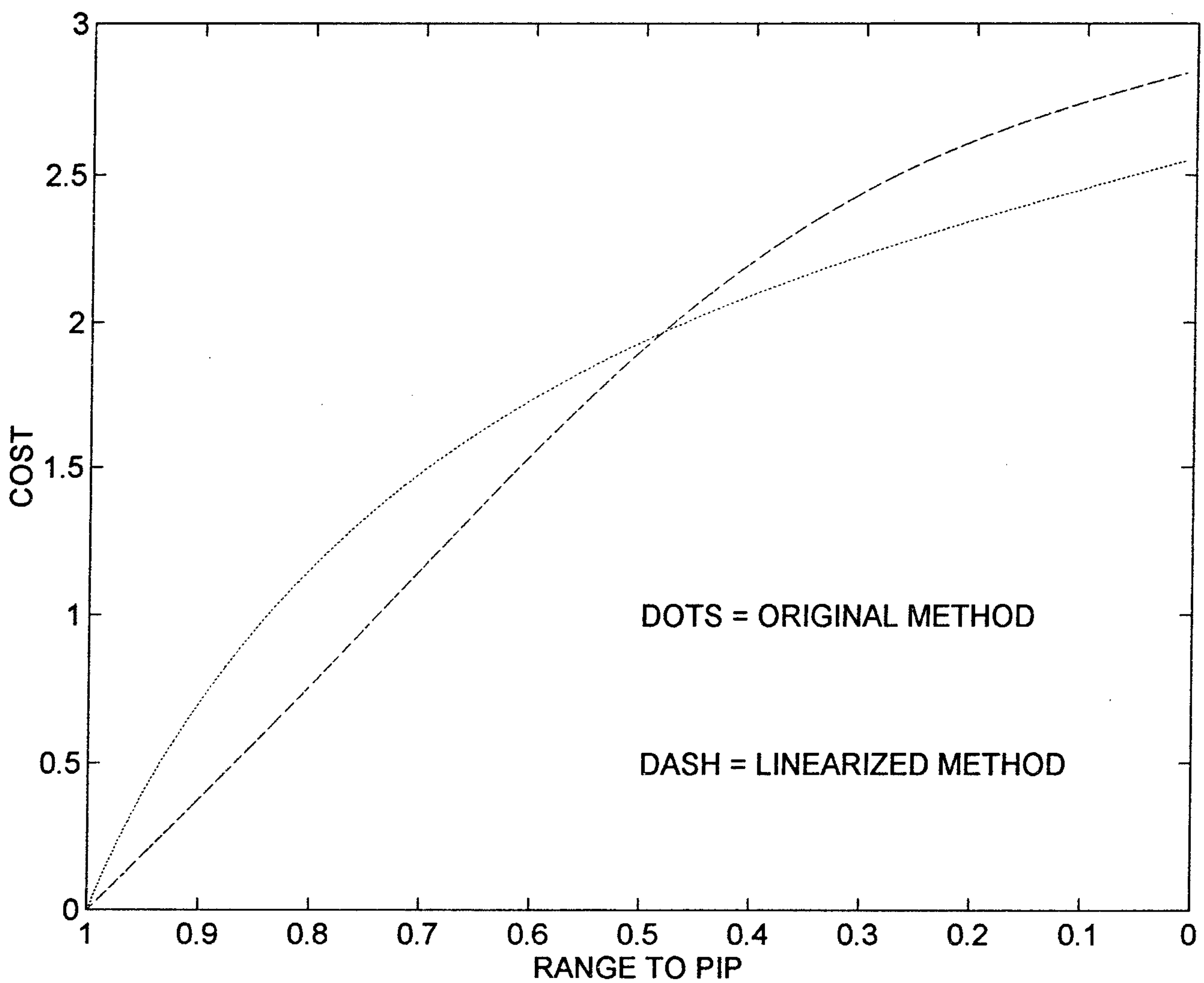

FIGURE 6-8. COST TRAJECTORY FOR $\gamma_0 = \delta_0 = \pi/3, \gamma_f =$ FREE

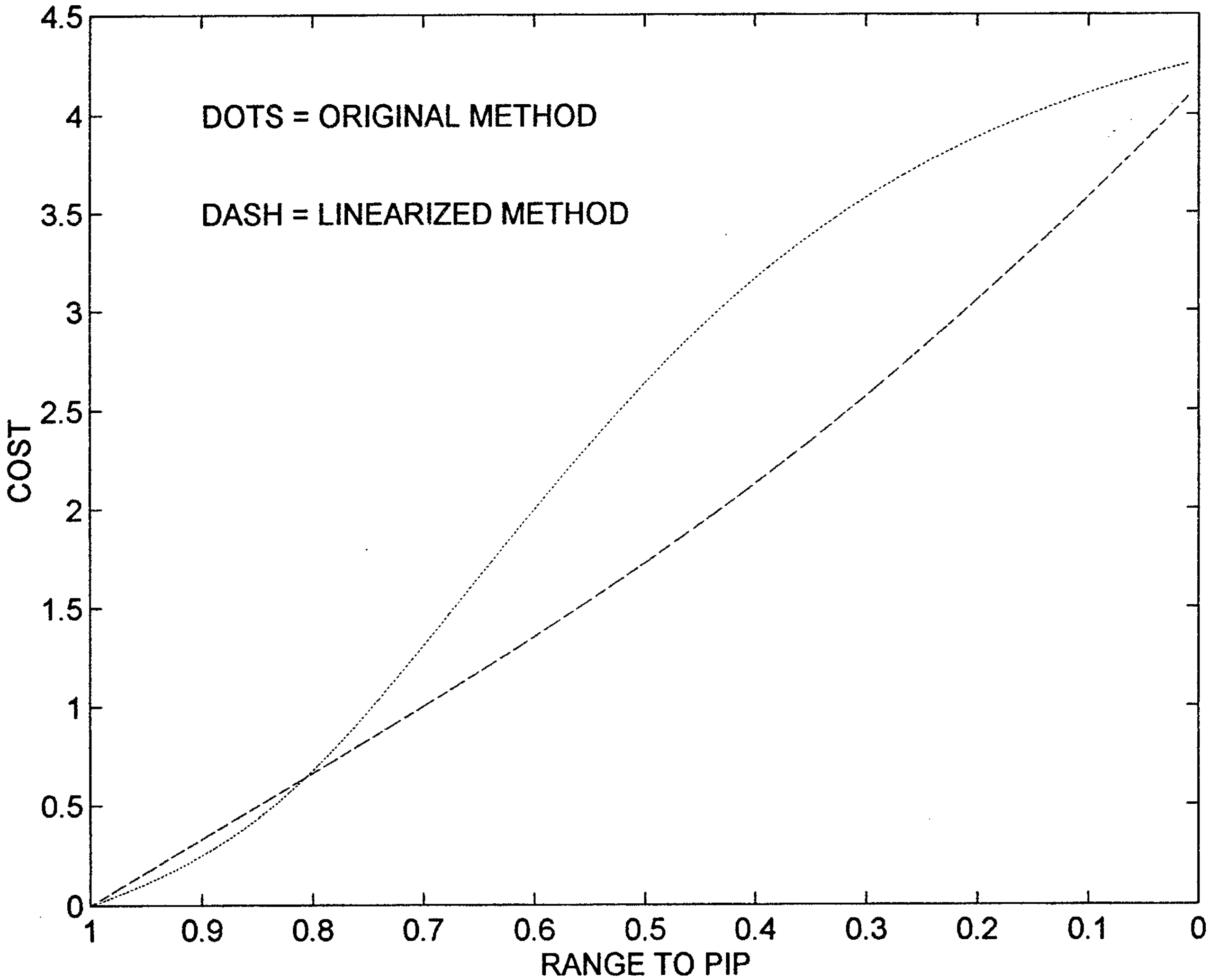

FIGURE 6-9. COST TRAJECTORY FOR $\gamma_0 = \delta_0 = \pi/3, \gamma_f = -\pi/2$

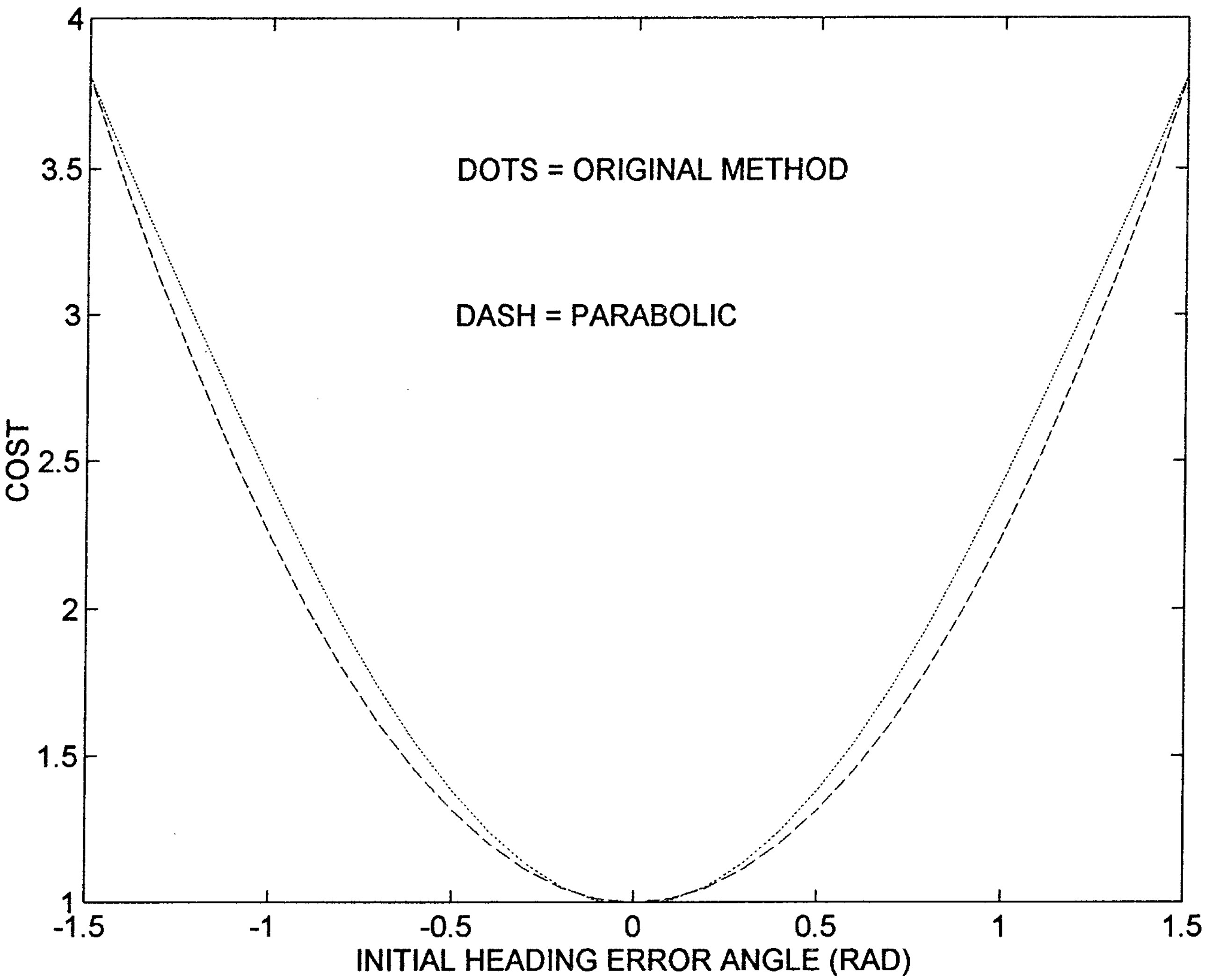

FIGURE 6-10. COST CURVE FOR $\gamma_0 = \delta_0 \in [-1.5, 1.5], \gamma_f =$ FREE

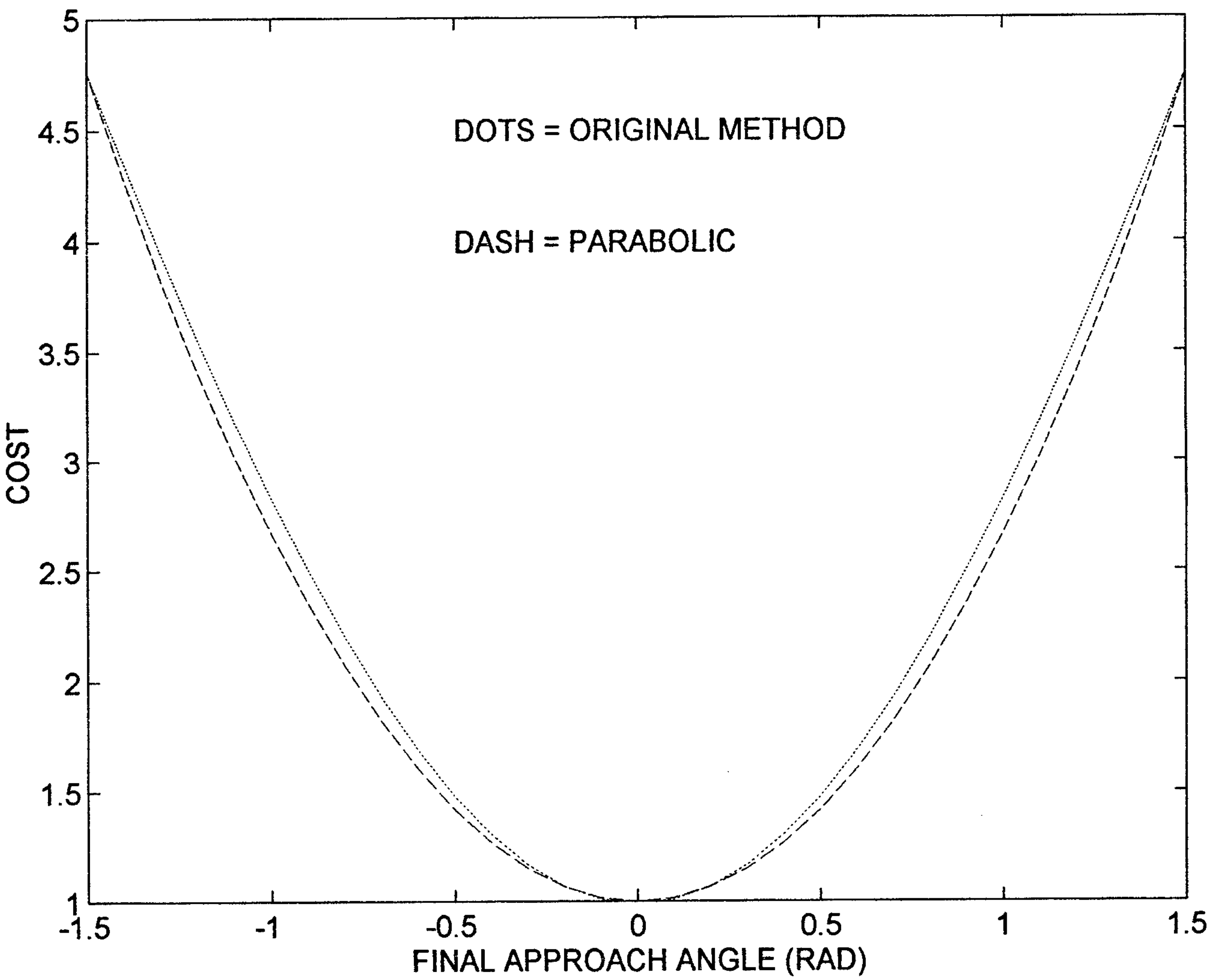

FIGURE 6-11. COST CURVE FOR $\gamma_0 = \delta_0 = 0, \gamma_f \in [-1.5, 1.5]$

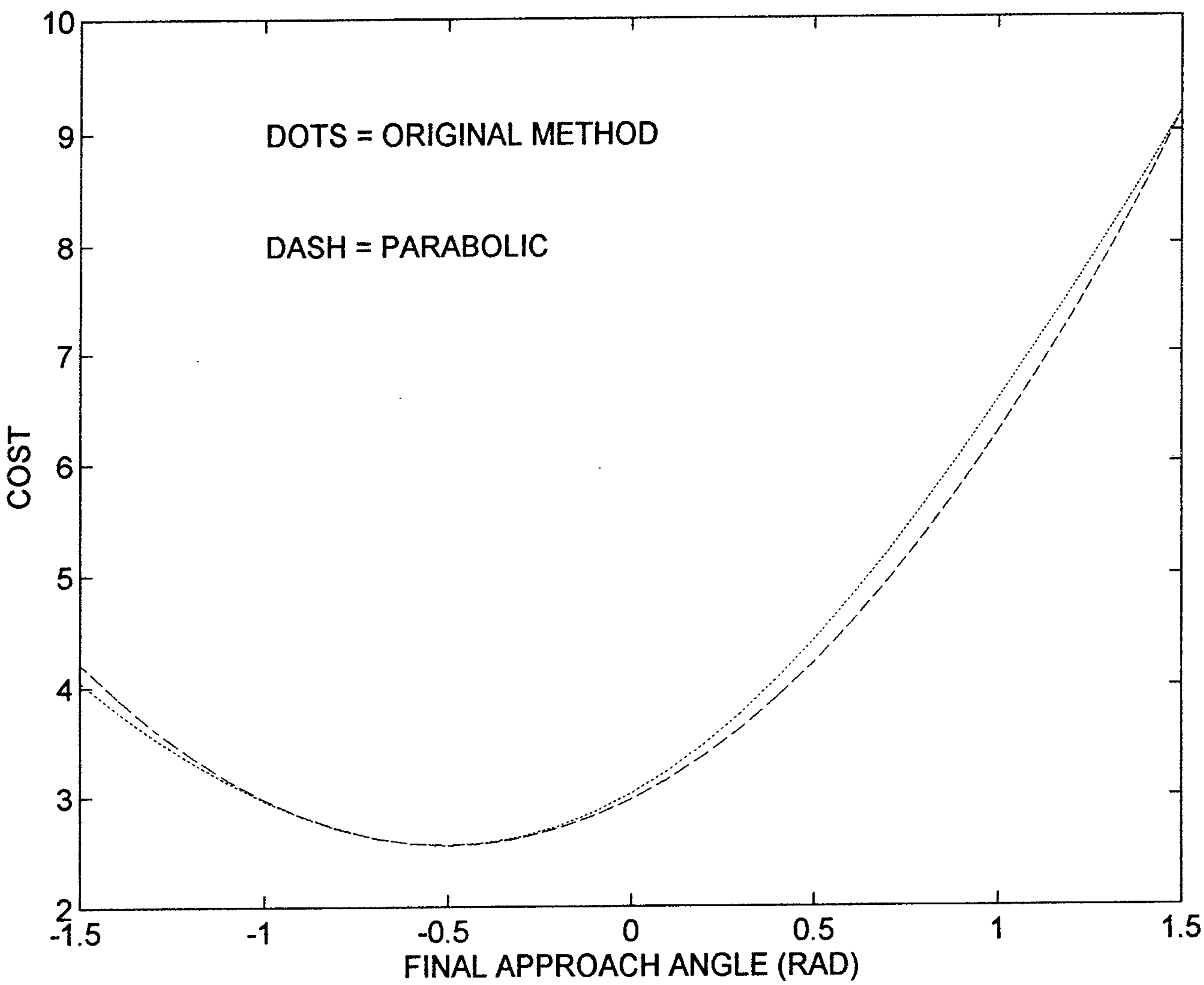

FIGURE 6-12. COST CURVE FOR $\gamma_0 = \delta_0 = \pi/3, \gamma_f \in [-1.5, 1.5]$

that computed at the beginning and end of a trajectory increment. For the original method, again the current values of the state variables, the downrange and altitude coordinates, and the range R to the PIP are assumed given. Then the optimal input, κ, to the plant equations (Equation (2.1)) is computed using the optimal feedback controller equations previously mentioned in Lin[1] and the plant is integrated. The downrange and altitude coordinates are incremented using the same procedure used with the linearized method. The original κ guidance method is coded in FORTRAN while the linearized κ guidance method is coded in MATLAB.

For either method, if the PIP were to change at some point during the generation of the trajectory, the new PIP would be used in the computation of the next trajectory increment, (if it were assumed that a missile had flown the trajectory up to that point) and the trajectory computation would proceed as described above. Now, suppose a missile is flying a midcourse trajectory with free final angle of approach. Let the PIP move, sometime during the course of the flight, in such a way that the final angle of approach is undesirable from an operational standpoint (e.g., seeker looking into the sun). With the linearized method described in this report, such a circumstance could be checked using the closed form solution for the velocity vector angle given by (5.32) and, if necessary, the rest of the midcourse trajectory could be recomputed using a specified fixed final angle of approach (e.g., not into the sun). With the original method, such a circumstance could be checked only by recomputing the whole trajectory.

CHAPTER 7

CONCLUSION

This report presents an alternate solution method to the κ guidance optimal control problem. The primary feature of this alternate method is that closed form solutions to the state variables have been computed for both the free terminal angle of approach case and the fixed terminal angle of approach case. Such solutions do not exist for the original method. The closed form solutions to the state variables for the linearized method established in this report may be advantageous in certain operational or simulation situations. For example, simulation studies may require less computing through the use of the closed form solutions. In operational situations requirements may be presented, for example, that certain velocity vector angles be avoided. The closed form solutions presented in this paper may be able to quickly determine if such requirements are met. It was seen in Chapter 6, the Numerical Examples chapter, that the cost curves can probably be functionalized. Finally, we remark that the technique of linearization with quadratic cost approximation may be of use as a general technique for computing suboptimal solutions to optimal control problems.

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APPENDIX A

STATE EQUATION AND COST FUNCTION DISCREPANCY

In this appendix, we will discuss the discrepancy of the state equations and cost function in this report with the state and cost function for kappa guidance given in Lin,[A-1] Gray-Hecht,[A-2] Ohlmeyer[A-3] and Serakos-Lin.[A-4] To make this appendix self-contained, we repeat the state equations and cost function as presented in this report. The state equations are

$$\frac{d}{dP}\begin{bmatrix}\gamma_1\\ \delta_1\end{bmatrix} = \begin{bmatrix}\kappa_1 \sec(\delta_1)\\ \kappa_1 \sec(\delta_1) + \tan(\delta_1)/(R_0 - P)\end{bmatrix}. \tag{A.1}$$

The cost function is

$$\mathcal{J}(\gamma_1, \delta_1, P, \kappa_1) = \int_0^{R_0} \left[(\kappa_1^2/2) + \omega^2\right] \sec(\delta_1)\, dP\ . \tag{A.2}$$

The state equations and cost function for kappa guidance as given by the above mentioned references are in terms of the range R (recall $R = R_0 - P$). The state equations as given by the references are

$$\frac{d}{dP}\begin{bmatrix}\gamma_2\\ \delta_2\end{bmatrix} = \begin{bmatrix}-\kappa_2 \sec(\delta_2)\\ -\kappa_2 \sec(\delta_2) - \tan(\delta_2)/R\end{bmatrix}. \tag{A.3}$$

The cost function as given by the references is

$$\mathcal{J}(\gamma_2, \delta_2, R, \kappa_2) = \int_0^{R_0} \left[(\kappa_2^2/2) + \omega^2\right] \sec(\delta_2)\, dR\ . \tag{A.4}$$

The "1" and "2" subscripts are intended to distinguish variables used in the two formulations thereby avoiding confusion. Substituting $R = R_0 - P$, $dR/dP = -1$ into Equation (A.1) results in

$$\frac{d}{dR}\begin{bmatrix}\gamma_1\\ \delta_1\end{bmatrix} = (-1)\cdot\frac{d}{dP}\begin{bmatrix}\gamma_1\\ \delta_1\end{bmatrix} = \begin{bmatrix}-\kappa_1 \sec(\delta_1)\\ -\kappa_1 \sec(\delta_1) - \tan(\delta_1)/R\end{bmatrix}, \tag{A.5}$$

which is identical to Equation (A.3). The same substitution in Equation (A.2) gives

$$\int_{P=0}^{P=R_0} \left[(\kappa_1^2/2) + \omega^2\right] \sec(\delta_1)\, dP = \int_{R=R_0}^{R=0} \left[(\kappa_1^2/2) + \omega^2\right] \sec(\delta_1)\,(-1) dR$$

which does not match with Equation (A.4). Treating P as a dummy variable in Equation (A.2) and R as a dummy variable in Equation (A.4), these two equations are identical.

The distinction between the first pair of equations and the second pair of equations is that "dP" is positive in both Equations (A.1) and (A.2) while "dR" is negative in Equation (A.3) and positive in Equation (A.4). To apply the minimum principle as it is given in Athans-Falb,[A-5] the differential of the independent variable must have the same sign in both the state equations and the cost function. Looking at the proof of the minimum principle in Athans-Falb,[A-5] page 309, Equation (5-503), an augmented differential equation is formed out of the integrand of the cost function and the state equations. This augmentation requires that the differential of the state equations and the cost function be the same, i.e, have the same sign. In other words, applying the minimum principle as stated in Athans-Falb,[A-5] to the optimal control problem given by Equations (A.3) and (A.4), and introducing a minus sign to Equation (A.4) so that the "$dR's$" line up, the cost function is negative. To illustrate this point further, consider the following example:

Example : Consider the system

$$\frac{d}{dt}\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}\cdot u \tag{A.6}$$

with cost function

$$\int_0^T \{(2+\frac{T^2}{2})\zeta_2^2 + 2T\zeta_2 u + \frac{T^2}{2}u^2\}dt\ . \tag{A.7}$$

The initial time is $t = T$ and the final time is $t = 0$. The initial condition is $(\zeta_1(T), \zeta_2(T)) = (\zeta_{1T}, \zeta_{2T})$ and the final condition is free. The differential "dt" is negative in Equation (A.6) and positive in Equation (A.7). Ignoring the difference in the sign of the differential and applying the minimum principle, we get that the Hamiltonian is

$$H = (2+\frac{T^2}{2})\zeta_2^2 + 2T\zeta_2 u + \frac{1}{2}T^2u^2 + p_1\cdot\zeta_2 + p_2\cdot u\ . \tag{A.8}$$

The adjoint equation is

$$\frac{d\overrightarrow{p}}{dt} = -\frac{\partial H}{\partial\overrightarrow{\zeta}} = -\begin{bmatrix}0\\ p_1 + (4+T^2)\zeta_2 + 2Tu\end{bmatrix}\ . \tag{A.9}$$

The terminal conditions for the adjoint variables are $(p_1(0), p_2(0)) = (0,0)$. The optimal control is

$$u^* = -\left(\frac{2\zeta_2}{T} + \frac{p_2}{T^2}\right)\ . \tag{A.10}$$

The Hamiltonian system is

$$\frac{d}{dt}\begin{bmatrix}\overrightarrow{\zeta}\\ \overrightarrow{p}\end{bmatrix} = \begin{bmatrix}W & -S\\ -Q & -W'\end{bmatrix}\begin{bmatrix}\overrightarrow{\zeta}\\ \overrightarrow{p}\end{bmatrix}, \tag{A.11}$$

where

$$W = \begin{bmatrix}0 & 1\\ 0 & -2/T\end{bmatrix},\ Q = \begin{bmatrix}0 & 0\\ 0 & T^2\end{bmatrix},\ \text{and } S = \begin{bmatrix}0 & 0\\ 0 & 1/T^2\end{bmatrix}\ . \tag{A.12}$$

The Riccati equation is

$$\frac{dK(t)}{dt} + KW + W'K - KSK + Q = 0\ . \tag{A.13}$$

The boundary condition is $K(0) = 0$. The solution to the Riccati equation is

$$K = \begin{bmatrix}0 & 0\\ 0 & \dfrac{t^3\sinh(t)}{\sinh(t) - t\cosh(t)}\end{bmatrix}\ . \tag{A.14}$$

Note that K is negative definite. The Hamilton-Jacobi equation is

$$\frac{\partial J}{\partial t} + \min_u H(\overrightarrow{\zeta}, \frac{\partial J}{\partial\overrightarrow{\zeta}}, u, t) = 0\ . \tag{A.15}$$

The reason we use the state and cost given by Equations (A.1) and (A.2) is because we want to use the minimum principle as it is given in Athans-Falb.[A-5]

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[A-3] Ernest J. Ohlmeyer, *Standard Missile Midcourse Guidance Laws*, NSWCDD Internal report, Code G23, Naval Surface Warfare Center, Dahlgren Division, Dahlgren, VA 22448, April 1994.

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APPENDIX B

TERMINAL COST ON INTERCEPT ANGLE

In this appendix, an intercept angle is specified by assigning a cost on any deviation from that intercept angle. The terminal cost constraint may be thought of as a "soft" constraint on the intercept angle. This is in contradistinction to the case where an intercept angle was fixed, which is a "hard" constraint.

The Hamiltonian system is

$$\frac{d}{dP}\begin{bmatrix} \overrightarrow{z} \\ \overrightarrow{p} \end{bmatrix} = \begin{bmatrix} W & -S \\ -Q & -W' \end{bmatrix}\begin{bmatrix} \overrightarrow{z} \\ \overrightarrow{p} \end{bmatrix} \tag{B.1}$$

where

$$W = \begin{bmatrix} 0 & 1 \\ 0 & 2/R \end{bmatrix}, \; Q = \begin{bmatrix} 0 & 0 \\ 0 & R^2\omega^2 \end{bmatrix}, \text{ and } S = \begin{bmatrix} 0 & 0 \\ 0 & 1/R^2 \end{bmatrix}. \tag{B.2}$$

The solution to the linear homogeneous equation, Equation (B.1), is

$$\begin{bmatrix} \overrightarrow{z} \\ \overrightarrow{p} \end{bmatrix}(P_1) = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}(P_1, P_2)\begin{bmatrix} \overrightarrow{z} \\ \overrightarrow{p} \end{bmatrix}(P_2), \tag{B.3}$$

where the Ω_{ij} are (unknown) 2×2 matrices. We set the terminal cost to be

$$K(\overrightarrow{x}) = \frac{1}{2}c_\gamma(x_1 - \gamma_f)^2 = \frac{1}{2}c_\gamma(z_1 - \tan^{-1}(Rz_2) + \gamma_f)^2 . \tag{B.4}$$

This is a quadratic cost, which is in keeping with our LQR setup. The transversality conditions for this terminal cost are

$$\begin{bmatrix} p_1(P = R_0) \\ p_2(P = R_0) \end{bmatrix} = \left.\frac{\partial K}{\partial \overrightarrow{z}}\right|_{P=R_0} = c_\gamma(z_1(P = R_0) + \gamma_f)\begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{B.5}$$

The known initial conditions on $\overrightarrow{z}(P = 0)$ (state variables) and Equation (B.5) give all the necessary boundary conditions for Equation (B.1). Two boundary conditions are on the state variables and two conditions are on the adjoint variables. Rewriting Equation (B.5),

$$\overrightarrow{p}(P = R_0) = c_\gamma\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} z_1(P = R_0) \\ z_2(P = R_0) \end{bmatrix} + c_\gamma\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cdot\gamma_f . \tag{B.6}$$

Setting $P_1 = 0$ and $P_2 = R$ and substituting this into Equation (B.3) gives

$$\left\{\Omega_{22}(R_0, P) - c_\gamma\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\Omega_{12}(R_0, P)\right\}\overrightarrow{p}(P)$$

$$= \left\{c_\gamma\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\Omega_{11}(R_0, P) - \Omega_{21}(R_0, P)\right\}\overrightarrow{z}(P) + c_\gamma\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cdot\gamma_f .$$

Thus,

$$\overrightarrow{p}(P) = K(P)\overrightarrow{z}(P) + \overrightarrow{g}(P)\cdot\gamma_f , \tag{B.7}$$

where we have set

$$K(P) = \left\{ \Omega_{22}(R_0, P) - c_\gamma \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Omega_{12}(R_0, P) \right\}^{-1} \cdot \left\{ c_\gamma \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Omega_{11}(R_0, P) - \Omega_{21}(R_0, P) \right\} , \tag{B.8}$$

and

$$\vec{g}(P) = \left\{ \Omega_{22}(R_0, P) - c_\gamma \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Omega_{12}(R_0, P) \right\}^{-1} \cdot c_\gamma \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \tag{B.9}$$

Here, $g(P)$ is a 2×1 column vector. We have that K satisfies the Riccati equation,

$$\frac{dK}{dP} + KW + W'K - KSK + Q = 0 , \tag{B.10}$$

and $\vec{g}$ satisfies

$$\frac{d\vec{g}}{dP} = (KS - W')\vec{g} . \tag{B.11}$$

From Equations (B.6) and (B.7), we have

$$K(R_0) = c_\gamma \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} , \text{ and } \vec{g}(R_0) = c_\gamma \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \tag{B.12}$$

These boundary conditions are independent of γ_f; hence, they may be solved once, numerically, and stored.

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